Pricing and Production Planning Under Supply Uncertainty

Burak Kazaz

Whitman School of Management, Syracuse University, 721 University Ave., Syracuse, NY 13244

(bkazaz@syr.edu, phone: 315-443-7381)

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Abstract

This paper addresses the problem of determining the sale price and the production quantity under supply uncertainty. The problem is commonly observed in the agricultural industry. We consider a firm that initially leases farm space in order to grow fruit. The realized amount of fruit supply fluctuates due to weather conditions, diseases, etc. At the end of the growing season, the firm makes two production decisions: the amount of realized supply to be converted to finished product and the amount of additional supply to purchase from other growers. However, the second opportunity to purchase from other growers occurs at a unit cost that depends on the realized supply. This is defined as the *yield-dependent purchasing cost*. Specifically, the lower the realized supply, the higher the unit purchasing cost.

Two modeling approaches are presented for the price-setting behavior. In the Early Pricing model, the sale price is determined when the leasing agreement is made, and in the Postponed Pricing model, it is set after observing the realized supply. The presence of the yield-dependent purchasing cost differentiates this work from studies that feature price-setting behavior under supply uncertainty. In traditional models where the unit purchasing cost is a constant, the firm experiences either the case of "complete backlogging" (all demand is satisfied) or "lost sales" (some of the demand is not satisfied), but not both. The yield-dependent cost, however, forces the firm to experience both cases simultaneously in this problem. The paper first identifies the optimal sale price and production decisions for the two variants of the problem with complete backlogging and lost sales, and later shows how these decisions differ when the firm utilizes the purchasing alternative with the yield-dependent cost. In the Early Pricing model, it proves that 1) the optimal stocking level is greater than or equal to its equivalent developed for the complete backlogging and lost sales variants, 2) the optimal sale price is greater than or equal to that of the lost sales variant, and is less than or equal to that of the complete backlogging variant, and 3) the purchasing alternative does not always reduce the firm's initial investment in the leased farm space. In the Postponed Pricing model, it shows that the optimal amount of farm space to be leased can be uniquely determined and that the contribution of the purchasing alternative is always positive.

Keywords: supply uncertainty, yield-dependent purchasing cost, pricing

1 Introduction and Literature Review

On December 2, 2006, *the New York Times* reported in the article "Nature Getting the Blame for Costly Orange Juice" that the average price of a gallon of orange juice had increased by 14%, going from $4.45 a year earlier to $5.09. The article states that the two largest orange juice producers in the US, Tropicana of Pepsi Co. and Minute Maid of Coca-Cola Company, warned consumers of more price increases in 2007 because of what is expected to be a lackluster orange crop. The orange supply in 2007 turned out to be the smallest in 17 years due to a summer where four hurricanes (Charley, Frances, Jeanne, and Wilma) crisscrossed the state of Florida, which satisfies more than 80% of the country's demand. On top of Florida's low orange supply, the state had to battle with citrus canker, the highly contagious bacterial disease that destroys citrus crops. On January 16, 2007, the Associated Press reported that four consecutive nights of freezing temperatures ruined approximately $1 billion worth of California's citrus, and further increased the concerns for the country's orange supply and its sale price.

The fluctuations in the orange supply exemplify how important it is for agricultural businesses to incorporate supply uncertainty into pricing and production planning decisions. This paper responds to this need by examining the combined decisions of price setting and production planning under supply uncertainty. As evidenced from the above examples, crop yield fluctuations have a significant impact on the profitability of agricultural businesses. Firms in the agricultural industry counter supply uncertainty by employing three methods. The first method corresponds to a common practice among orange juice producers in the US, olive oil producers in Turkey, and some of the largest wine producers throughout the world. These producers lease farm space in order to grow fruit in anticipation of reducing their future purchasing costs. The lease is determined by the number of trees, and the producer incurs the cost of growing the fruit and maintaining the farm space (which includes pruning, stem cutting, weed control, and insect and disease management). However, this strategy forces the producer to deal with the additional risk of supply uncertainty. The production of olive oil, in particular, is a risky investment because olive trees bear fruit only once in two years. The realized supply at the end of the two-year-long growing season can fluctuate significantly due to weather conditions, diseases, etc. Leasing farm space is also widely practiced in the wine industry. One executive of a large wine producer and distributor has described the benefit of leasing farm space as reducing the risk associated with a lower return on equity and the perceived value of the firm in financial markets.

The second method of battling supply uncertainty involves the second chance opportunity of purchasing additional fruit from other growers when the realized supply is insufficient. However, as described in the above example about the orange supply, the unit purchasing cost (of oranges or olives) changes with the amount of realized supply. Specifically, the lower the realized supply, the higher the unit purchasing cost. The Ayvalık Chamber of Commerce in Turkey, the region that provides more than 70% of the country's production of olive oil, has reported how the crop yield fluctuations in the Turkish olive oil industry influences the purchasing cost of olives, and Kazaz (2004) has defined the relationship between the unit purchasing cost and the crop yield realization as "yield-dependent." Similar observations can be made in the wine industry. The labeling requirements with references to the growing region (e.g. Sonoma Country, Napa Valley) and the year of the vintage force this industry to operate under a yield-dependent purchasing cost, where the unit cost is strongly related to the amount of the crop collected in the region in a specific year. In the case of Australian wine grapes, the purchasing cost increased more than five times from a year before based on the realized supply. Due to an oversupply in 2006, the purchasing price of wine grapes were reported to be averaging approximately $100 per ton, going below the cost of growing the fruit (ABC News Online, http://www.abc.net.au/news/newsitems/200604/s1615457.htm). *The New York Times* reported in the article "A Drought in Australia" on April 17, 2008 that a long drought in 2007, however, cut the country's wine grape supply by more than half, causing the purchasing cost of wine grapes to rise significantly, ranging between $500 to $3000 per ton. The end result of these observations is that the yield-dependent cost of purchasing supply needs to be incorporated in the pricing and production planning problem of the leading agricultural businesses. As will be shown later, the yield-dependent unit purchasing cost has a significant impact on the optimal decisions regarding the sale price and the production quantity. When compared, the expected value of the yield-dependent purchasing cost over supply uncertainty is typically higher than the expected cost of growing the fruit. This fact provides another justification for the firm to lease farm space. It should also be emphasized that, unlike the practices observed in the manufacturing industry, agricultural businesses cannot use inventories strategically to battle supply uncertainty. For example, the olive oil producer needs to press its olives within 48 hours of collection in order to achieve the highest quality of its final product. As a result, the oil producer cannot hold inventories of olives for future production of olive oil. Furthermore, olive oil that is stored longer than two years is considered to develop an acidic taste. Similar observations can be made for the production of fresh orange juice and wine. Freshly squeezed orange juice has a superior taste than the juice obtained from oranges stored in inventory. Therefore, high quality orange juice is obtained by pressing the oranges immediately after the collection. In wine production, grapes are also pressed immediately after their collection. Blackburn and Scudder (2008) provides other examples of deteriorating product quality in fresh produce (e.g. melons and sweet corn). Thus, like most perishable goods, the problem for agricultural businesses can be modeled in a single-period context.

The third method for battling uncertainties deals with influencing the demand by appropriately selecting the sale price of the final product. As exemplified by orange juice producers, setting prices can be an important tool for managing supply uncertainty. The price-setting behavior of the agricultural firm is examined with two different models in this paper. In the Early Pricing model, the sale price is determined at the same time when the leasing agreement is made. In the Postponed Pricing model, it is determined after supply uncertainty is resolved.

The modeling approach consists of two-stage stochastic programs with recourse, where the growing season of the fruit corresponds to the first stage of the model. In this stage, the producer determines the amount of farm space to be leased that maximizes the expected profit under supply uncertainty. The amount of farm space leased can be perceived as the maximum amount of production attained under the ideal growing condition. At the end of the growing season, the producer observes only a proportion of this amount. The selling season corresponds to the second stage of the model. In this stage, the producer makes two decisions in order to maximize its profit. The first decision is the amount of realized supply (from internally grown fruit through the leased farm space) to be converted to the final product. The second decision is the amount of additional fruit to be purchased from other growers for the production of the final product. In addition to these production-related decisions, the firm also determines the sale price. In the Early Pricing model, the price is determined in the first stage before realizing supply, and in the Postponed Pricing model, it is determined in the second stage after supply is revealed. While the Early Pricing model complements earlier research and provides a good benchmark for the analysis presented here, the Postponed Pricing model represents the operating conditions in most agricultural industries (e.g. olive oil and orange juice). The analysis provides the firm's optimal choices under two different demand functions: linear and iso-elastic in selling price.

There are several distinguishing factors that separate our work from earlier studies. Earlier research that examined production planning problems under supply uncertainty (as yield uncertainty) have considered prices to be exogenous. The body of literature that incorporates the price-setting behavior of a monopolistic firm into the decision making process under supply uncertainty is limited. This paper advances these works by investigating the impact of the yield-dependent purchasing cost on pricing and production decisions under supply uncertainty. It should be highlighted here that the analysis regarding the yield-dependent purchasing cost is different than that of the convex or concave costs, as used in the traditional Newsvendor Problems under demand uncertainty (see Porteus (1990) for a discussion), because the yield-dependent unit purchasing cost need not be convex or concave. Since the value of the unit purchasing cost is unknown at the beginning of the planning horizon, the purchasing cost at time zero can be perceived as random. When the unit purchasing cost is defined as a deterministic parameter, the firm's preference at every realization of supply that is less than the demand in the Early Pricing model would be either to not purchase additional supplies and lose sales or purchase additional supplies in order to satisfy all demand. We refer to the first action as lost sales and the latter as "complete backlogging" because it resembles the case of satisfying the demand ultimately. An important difference with the yield-dependent unit purchasing cost is that lost sales is optimal for some realization of supply less than the demand and complete backlogging is optimal for other realizations of supply less than the demand. This further complicates the solution approach necessary to obtain the optimal policy. The third factor corresponds to the presence of recourse actions. After supply uncertainty is resolved, the recourse feature of our problem provides the firm with increased flexibility. For example, in addition to the option of purchasing additional supplies when the realized yield is low, the firm may elect to not sell all supply in the market when the realized yield is high.

The paper is organized in the following order: A review of the literature is provided in Section 2. The Early Pricing model is presented in Section 3, and the Postponed Pricing model in Section 4. Section 5 provides the conclusions, managerial insights, and future research directions. Numerical illustrations based on data regarding the leading olive oil producers in Turkey (provided by the Ayvalık Chamber of Commerce) are presented in Appendix A. An online addendum, Appendix B, contains all proofs and derivations.

2 Literature Review

The problem of production planning under supply uncertainty has received considerable attention. Price is commonly assumed to be exogenous. Yano and Lee (1995) provide an extensive review of production and inventory problems under yield uncertainty, which is the foundation for supply uncertainty. Models that consider production capabilities in a single and multiple period(s) can be found in Gerchak et al. (1988), Gerchak (1992), Henig and Gerchak (1990), Henig and Levin (1992) and Shih (1980). Gerchak et al. (1988) and Henig and Gerchak (1990) prove that the optimal production quantity does not depend on the yield distribution in a periodic review inventory problem. Henig and Levin (1992) determine the optimal order quantity with the choice of the vendor and the quantity to be delivered to customers. Hsu and Bassok (1999) and Bitran and Silgert (1994) identify the amount of optimal production with the availability of downward substitution. Bollapragada and Morton (1999) describe efficient myopic heuristics for periodic review inventory problems. The problem of production planning under random yield is also considered in assembly lines (see Gerchak et al. (1994)) and in N-stage serial systems (see Lee and Yano (1988)). Grosfeld-Nir and Gerchak (2002) provide an extensive review of the literature that studies multiple lot sizing decisions under random supply and demand in make-to-order systems. Rajaram and Karmarkar (2002) consider the issue of supply uncertainty in the process industry. Galbreth and Blackburn (2006) extend this work in the context of remanufacturing.

The second opportunity to obtain additional fruit resembles the setting in Jones et al. (2001). In their paper, the hybrid seed corn producer gets a second chance of production in a different region of the world and experiences supply uncertainty. Our problem differs from the one presented in Jones et al. (2001) in two ways: 1) the sale price is endogenous to the model, therefore the producer uses the sale price as a mechanism to hedge against fluctuating supply, 2) when the producer purchases additional fruit from other growers she does not experience another supply uncertainty. A similar setting with one unreliable and one reliable supply source is also examined in Kazaz (2004), Tomlin (2006), Tomlin and Snyder (2006); and the scenario with multiple unreliable suppliers is investigated in Tomlin and Wang (2005), Dada et al. (2006), and Federgruen and Nang (2008). However, these papers consider only exogenous selling price, and are not concerned with the insights that come as a result of the price-setting behavior.

Price-setting behavior has been widely studied under demand uncertainty (see Petruzzi and Dada (1999, 2001), Federgruen and Heching (1999, 2002), Boyacı et al. (2006), Kocabıyıkoglu and Popescu (2007)), however, it has not been examined extensively with regard to supply uncertainty. Li and Zheng (2006) is the first study to consider endogenous pricing for inventory replenishment under supply and demand uncertainty. While their model resembles only one setting of this paper, the Early Pricing model with complete backlogging, it differs in three ways. First, in their model, supply (and demand) uncertainty are realized at the same time, whereas in our model supply uncertainty is revealed first, giving the opportunity to the firm to adjust its production decisions. The consequence of this difference is that their model does not provide a recourse action. Secondly, their model corresponds to the complete backlogging variant of our problem where the firm is required to satisfy all demand. This feature enables them to obtain unique optimal solutions without enforcing a restriction on the probability density function (pdf) of supply (and demand) uncertainty. In our problem, both backlogging and lost sales are possible depending on realized supply. As will be shown later, when the scenario of lost sales is allowed in the problem, however, it becomes necessary to enforce restrictions on the pdf of supply uncertainty in order to warrant the uniqueness of the optimal policy. Third, while their model utilizes the second opportunity to purchase additional supplies after supply and demand are realized, their purchasing cost is static (i.e., independent of the yield), and our purchasing cost changes with the realized supply. Both technical and managerial results depart significantly in the presence of the yield-dependent purchasing cost.

Tang and Yin (2007) also provide a comparison of early and postponed pricing under supply uncertainty with linearly decreasing demand. Similar to our findings, they also conclude that the optimal sale price in the Early Pricing Model is higher than its deterministic equivalent. Supply uncertainty in their paper is restricted to a discrete uniform distribution, and therefore, their results are not generalizable. This paper extends their work by incorporating continuous and arbitrary distributions that define supply uncertainty and by featuring the yield-dependent purchasing cost.

Tomlin and Wang (2006) consider price-setting behavior under supply uncertainty for a firm that produces multiple products with downward substitution. However, their prices are market-clearing prices as uncertainty is resolved before setting prices in their model; thus, the insight stemming from the price-setting behavior is limited.

3 The Early Pricing Model

This section presents the Early Pricing model and the impact of supply uncertainty caused by the fluctuations in yield. The model uses a two-stage stochastic program where uncertainty stems from the fluctuations in the realized supply. At the beginning of a planning horizon (corresponding to the first stage), the firm leases Q units of farm space and pays cQ, where c is the unit cost of leasing. The decision variable Q can be perceived as the planned amount of production, however, the firm receives only a proportion of this target production. The yield fraction is denoted by u, a random variable that has $g(u)$ as its pdf and $G(u)$ as its cumulative distribution function (cdf) defined on a support [0,1] with the expected value of $\overline{u}$. Thus, Qu is the realized supply at the end of the growing season for the agricultural firm. In the second stage of the model, corresponding to the selling season, the firm makes two production decisions: q_1 is the amount of final product produced from the internally grown supply (through the leased farm space), and q_2 is the amount of final product produced from an externally purchased supply. The first production decision, q_1, cannot exceed the realized supply, i.e., $q_1 \leq Qu$. The second production decision refers to the second chance opportunity to purchase additional supplies from other growers. In the case of an olive oil producing firm, for example, the company can decide to purchase more olives to be converted to olive oil, however, the unit purchasing cost of olives at this stage depends on the realization of crop yield. As stated earlier, the unit cost corresponding to the second chance of purchasing additional supplies is yield-dependent, and is defined as $c_2(u)$, where the unit cost is a decreasing function of u. The firm pays a unit pressing cost c_p for both internally grown and externally purchased supply in order to obtain the final product. At the end of the growing season, if there is any unused supply, i.e., $q_1 < Qu$, the remaining supply is salvaged at a unit revenue of h_1, where $h_1 < c_2(u=1)$, and $h_1 < \frac{c}{\overline{u}}$.

The firm's choice of the sale price, denoted by p, influences its demand. In order to highlight the impact of supply uncertainty in this section, demand is defined as deterministic but price-sensitive. The extension that incorporates demand uncertainty is discussed in Section 5. We consider two forms of price-sensitive demand. In the first, demand decreases linearly in the sale price and is defined as:

$$D(p) = a - bp \text{ where } a, b > 0. \tag{1}$$

In the second, we consider the iso-elastic demand, where the demand curve is defined as:

$$D(p) = ap^{-b} \text{ where } a > 0 \text{ and } b > 1. \tag{2}$$

The price elasticity of the demand function is represented with $\in(p) = \frac{-p\left(\frac{\partial D(p)}{\partial p}\right)}{D(p)}$. The results obtained under supply uncertainty are not influenced qualitatively by the definition of the demand curve, however, the exact values for the optimal decisions vary depending on whether a linear or an iso-elastic demand function is used.

In the first stage of the Early Pricing model (denoted by EP), the firm determines the sale price p and the target amount of production Q that maximizes the expected profit $E[P_E(p,Q)]$.

$$(EP): \max_{(p,Q\geq 0)} E[P_E(p,Q)] = -cQ + \int_0^1 PA(p,Q,u)\, g(u)\, du \tag{3}$$

where $PA(p,Q,u)$ is the recourse (return) function obtained from leasing Q units of farm space, however, observing a yield of only Qu units when the sale price is p.

In the second stage after observing the realized supply of Qu, the firm determines q_1 and q_2, corresponding to the amount of final products obtained through internally grown and externally purchased supplies, respectively. Thus, the second-stage objective function can be written as follows:

$$PA(p,Q,u) = \max_{(q_1,q_2\geq 0)} \pi(q_1,q_2 \mid p,Q,u) = \left\{ \begin{array}{c} p\min\{(q_1+q_2), D(p)\} \\ -c_p(q_1+q_2) - c_2(u)\, q_2 + h_1(Qu - q_1)^+ \end{array} \right\} \tag{4}$$

s.t.

$$q_1 < Qu \tag{5}$$

It should be noted here that the sale price is determined in the first stage, and therefore the demand denoted by $D(p)$ is already set from the first-stage decisions. The following proposition shows that the optimal choices in the second stage can be categorized in one of three potentially optimal scenarios.

Proposition 1 *For a given sale price p ($p > c_p + h_1$) and realized supply Qu, the optimal production decisions that maximize (4) are as follows:*

$$(q_1^*, q_2^*) = \left\{ \begin{array}{ll} (Qu, 0) & \text{when } Qu \leq D(p) \text{ and } c_2(u) > p - c_p \\ (Qu, D(p) - Qu) & \text{when } Qu \leq D(p) \text{ and } c_2(u) \leq p - c_p \\ (D(p), 0) & \text{when } Qu > D(p) \end{array} \right\}. \tag{6}$$

The above proposition highlights that the optimal production decisions in the second stage can be classified in three scenarios. Let us classify them as low, intermediate, and excess supply, respectively. Under the low supply scenario, the realized supply is less than the demand, i.e., $Qu \leq D(p)$, and the firm converts its entire crop to the final product, thus we have $q_1^* = Qu$. It should be observed that the firm has the flexibility to obtain more supplies when the realized supply is less than the demand (when $Qu \leq D(p)$), however, this decision depends on the value of the unit purchasing cost $c_2(u)$. The firm exercises this option (and purchases more supplies from other growers) only when $c_2(u) \leq p - c_p$. Because $c_2(u) > p - c_p$ in the low supply scenario, the firm does not purchase additional supplies despite the fact that demand is not completely satisfied. Under the intermediate supply scenario, the realized supply is again less than the demand, $Qu \leq D(p)$, however, the unit purchasing cost is less than the return, i.e., $c_2(u) \leq p - c_p$, and purchasing additional supplies is a viable option. In this scenario, the firm converts all of its realized crop and purchases more supplies in order to increase its total production to the level of demand. Under the excess supply scenario, the realized supply is higher than the demand, $Qu > D(p)$, and the firm converts only a proportion of its crop to the final product, equalling the production and the demand, i.e., $q_1^* = D(p)$. The excess supply corresponding to $Qu - D(p)$ is salvaged at a unit revenue of h_1. As a result of these optimal production choices in each scenario, the recourse function can be written as follows:

$$PA(p,Q,u) = \left\{ \begin{array}{ll} (p-c_p)Qu & \text{when } Qu \leq D(p) \text{ and } c_2(u) > p - c_p \\ (p-c_p)D(p) - c_2(u)(D(p) - Qu) & \text{when } Qu \leq D(p) \text{ and } c_2(u) \leq p - c_p \\ (p-c_p)D(p) + h_1(Qu - D(p)) & \text{when } Qu > D(p) \end{array} \right\}. \tag{7}$$

Substituting the above recourse function into the first-stage objective function in (3) and rearranging the terms provide the following expression:

$$E[P_E(p,Q)] = \left(p - c_p - \frac{c}{\overline{u}}\right) D(p) - E[underage_S] - E[overage_S] \tag{8}$$

where

$$\begin{aligned} E[underage_S] \quad = \quad & \left(p - c_p - \frac{c}{\overline{u}}\right) \int_0^{\max\left\{0, c_2^{-1}(p-c_p)\right\}} (D(p) - Qu)\, g(u)\, du \\ & + \int_{\max\left\{0, c_2^{-1}(p-c_p)\right\}}^{D(p)/Q} \left(c_2(u) - \frac{c}{\overline{u}}\right) (D(p) - Qu)\, g(u)\, du \end{aligned} \tag{9}$$

is the expected underage cost of supply uncertainty, and

$$E[overage_S] = \left(\frac{c}{\overline{u}} - h_1\right) \int_{D(p)/Q}^{1} (Qu - D(p))\, g(u)\, du \tag{10}$$

is the expected overage cost of supply uncertainty. The first term in (8), $\left(p - c_p - \frac{c}{\overline{u}}\right) D(p)$, is equivalent to the profit that is free from supply uncertainty; specifically, this is the revenue when the effect of supply uncertainty is discounted from the model and the cost of growing the fruit is less expensive than the (expected) cost of purchasing. The next two terms in (8) are the expected underage and the expected overage costs of supply uncertainty. Examining each term in depth provides useful managerial insight. The expected underage cost of supply uncertainty expressed in (9) has two integral terms where one represents the case of lost sales, and the other represents the incremental cost from the purchasing alternative. The first integral term of the underage cost, $\left(p - c_p - \frac{c}{\overline{u}}\right) \int_0^{\max\left\{0, c_2^{-1}(p-c_p)\right\}} [(D(p) - Qu)]\, g(u)\, du$, is equivalent to the foregone profits because of not having sufficient supplies due to a low crop yield. In this case, the yield is so low that the yield-dependent purchasing cost is higher than the revenue less the processing cost of the fruit. Therefore, the firm prefers not to buy more supplies, and does not fulfill its market demand completely. Thus, for each unit of demand that the firm cannot fulfill, it pays a cost of $\left(p - c_p - \frac{c}{\overline{u}}\right)$. The second integral term in the expected underage cost of supply uncertainty, $\int_{\max\left\{0,c_2^{-1}(p-c_p)\right\}}^{D(p)/Q}\left(c_2\left(u\right)-\frac{c}{\overline{u}}\right)\left(D\left(p\right)-Qu\right)g\left(u\right)du$, corresponds to the additional cost incurred through the option of purchasing. This term resembles the backlogging cost, however, the penalty cost of backorders changes with the realized value of the crop yield. It should be observed that $c_2\left(u\right)-\frac{c}{\overline{u}}$ is the additional cost for each unit of demand satisfied through purchasing from other growers rather than growing the fruit. A unique feature of the model in this paper can be easily observed in the expected cost of underage. In traditional papers investigating the joint pricing and production decisions under demand uncertainty, for example, the expected profit includes a term for either lost sales or backorders. However, both lost sales and backlogging terms are observed simultaneously under the same scenario (when the realized supply is less than the demand) in this problem. Finally, the expected overage cost of supply uncertainty is expressed as in (10), and the firm has the cost of growing the fruit $\frac{c}{\overline{u}}$ less the salvage revenue earned, h_1, for each unit of supply that exceeds the demand.

To summarize the above discussion, the first-stage objective function has three components: profit that is free of supply risk, the expected underage cost, and the expected overage cost of supply uncertainty. Analyzing these three terms in depth provides insight into the optimal choices for the sale price and production quantity. In order to establish a basis for comparison, we begin our discussion with the case of the deterministic supply.

3.1 Deterministic Supply

In the deterministic supply setting, we consider the pdf of supply uncertainty defined as $g(u)=1$ for $u=\overline{u}$ and $g(u)=0$ for all u such that $0\leq u\neq\overline{u}\leq 1$. In this setting, the firm uses either the option of growing the fruit or the option of purchasing it, but not both, in its optimal solution. When the cost of growing the fruit is lower than the cost of purchasing at $u=\overline{u}$, the firm does not utilize the purchasing option, and $q_1^*>0$ and $q_2^*=0$. However, when the purchasing cost of the fruit is lower than the expected growing cost, the firm does not utilize the growing option (i.e., $Q^*=q_1^*=0$), and relies solely on the purchasing alternative. In this case, the firm purchases an amount equivalent to its demand, and $q_2^*=D\left(p^*\right)>0$.

In the growing alternative of the deterministic supply analysis, the cost of acquiring one unit becomes $\frac{c}{\overline{u}}$. Denoting the optimal decisions for the sale price and and production quantities with p_g^0, $Q_g^0, q_{1g}^0, q_{2g}^0$, one can easily see that $Q_g^0=\frac{q_{1g}^0}{\overline{u}}=\frac{D\left(p_g^0\right)}{\overline{u}}$ in the deterministic case. Using the linearly decreasing demand in (1), the optimal sale price and production decisions that are free of supply uncertainty are as follows:

$$p_g^0 = \frac{a+b\left(c_p+\frac{c}{\overline{u}}\right)}{2b}; \tag{11}$$

$$\left(Q_g^0, q_{1g}^0, q_{2g}^0\right) = \left(\frac{a-b\left(c_p+\frac{c}{\overline{u}}\right)}{2\overline{u}}, \frac{a-b\left(c_p+\frac{c}{\overline{u}}\right)}{2}, 0\right). \tag{12}$$

Similarly, when the iso-elastic demand function in (2) is used, the optimal decisions are:

$$p_g^0 = \frac{b}{(b-1)}\left(c_p+\frac{c}{\overline{u}}\right); \tag{13}$$

$$\left(Q_g^0, q_{1g}^0, q_{2g}^0\right) = \left(\frac{a}{\overline{u}}\left(\frac{b}{(b-1)}\left(c_p+\frac{c}{\overline{u}}\right)\right)^{-b}, a\left(\frac{b}{(b-1)}\left(c_p+\frac{c}{\overline{u}}\right)\right)^{-b}, 0\right). \tag{14}$$

In the purchasing alternative of the deterministic supply analysis, the unit cost of acquiring the fruit can be described in two different ways. Note that one can express the unit cost of purchasing the fruit at $u = \overline{u}$ with $c_2(\overline{u})$. When the unit purchasing cost is described as $c_2(\overline{u})$, we denote the optimal decisions for the sale price and production quantities as p_p^0,Q_p^0, q_{1p}^0, q_{2p}^0. Using the linearly decreasing demand in (1), the optimal decisions are:

$$p_p^0 = \frac{a + b(c_p + c_2(\overline{u}))}{2b}; \tag{15}$$

$$\left(Q_p^0, q_{1p}^0, q_{2p}^0\right) = \left(\frac{a - b(c_p + c_2(\overline{u}))}{2\overline{u}}, \frac{a - b(c_p + c_2(\overline{u}))}{2}, 0\right). \tag{16}$$

Similarly, when the iso-elastic demand function in (2) is used, the optimal decisions are:

$$p_p^0 = \frac{b}{(b-1)}(c_p + c_2(\overline{u})); \tag{17}$$

$$\left(Q_p^0, q_{1p}^0, q_{2p}^0\right) = \left(0, 0, a\left(\frac{b}{(b-1)}(c_p + c_2(\overline{u}))\right)^{-b}\right). \tag{18}$$

It is important to highlight that the unit purchasing cost $c_2(\overline{u})$ is not necessarily equal to the expected unit purchasing cost $E(c_2(u)) = \int_0^1 c_2(u)\, g(u)\, du$. Moreover, depending on the form of the unit purchasing cost function $c_2(\overline{u})$ can be less than or greater than $E(c_2(u))$. Therefore, in the deterministic supply analysis, it is insightful to prescribe the optimal sale price and production decisions (denoted by p_p^{00} and $Q_p^{00}, q_{1p}^{00}, q_{2p}^{00}$) by substituting the expected value of the purchasing cost. Using the linearly decreasing demand in (1), the optimal decisions are:

$$p_p^{00} = \frac{a + b(c_p + E(c_2(u)))}{2b}; \tag{19}$$

$$\left(Q_p^{00}, q_{1p}^{00}, q_{2p}^{00}\right) = \left(\frac{a - b(c_p + E(c_2(u)))}{2\overline{u}}, \frac{a - b(c_p + E(c_2(u)))}{2}, 0\right). \tag{20}$$

Similarly, when the iso-elastic demand function in (2) is used, the optimal decisions are:

$$p_p^{00} = \frac{b}{(b-1)}(c_p + E(c_2(u))); \tag{21}$$

$$\left(Q_p^{00}, q_{1p}^{00}, q_{2p}^{00}\right) = \left(0, 0, a\left(\frac{b}{(b-1)}(c_p + E(c_2(u)))\right)^{-b}\right). \tag{22}$$

Note that $p_p^0 < p_p^{00}$ when $c_2(\overline{u}) < E(c_2(u))$, and $p_p^0 > p_p^{00}$ when $c_2(\overline{u}) > E(c_2(u))$. These two price expressions for the deterministic setting are insightful in the analysis pertaining to stochastic supply. In order to represent the more realistic and interesting case, we concentrate on the case when $\{c_2(\overline{u}), E(c_2(u))\} > \frac{c}{\overline{u}}$ where the firm has a higher unit purchasing cost than the expected cost of growing the fruit.

It should be pointed out here that it is the supply uncertainty that causes the firm to use both the growing and the purchasing options simultaneously; thus, the firm can have q_1^* and q_2^* with positive values at the same time only in the stochastic supply setting, but not in the deterministic setting. Before proceeding with the analysis of supply uncertainty, we provide a transformation of the decision variable, $z \equiv \frac{D(p)}{Q}$, and express the first-stage objective function as follows:

$$E[P_E(p, z)] = \left(\frac{D(p)}{z}\right) L(p, z) \tag{23}$$

where

$$
\begin{aligned}
L(p,z) &= \left(p - c_p - \frac{c}{\overline{u}}\right) z \\
&\quad - \left(p - c_p - \frac{c}{\overline{u}}\right) \int_0^{\max\left\{0, c_2^{-1}(p-c_p)\right\}} \left[(z-u)\right] g(u)\, du - \int_{\max\left\{0, c_2^{-1}(p-c_p)\right\}}^{z} \left[\left(c_2(u) - \frac{c}{\overline{u}}\right)(z-u)\right] g(u)\, du \\
&\quad - \left(\frac{c}{\overline{u}} - h_1\right) \int_z^1 (u-z)\, g(u)\, du. \qquad (24)
\end{aligned}
$$

In order to have a comprehensive perspective regarding the influence of the yield-dependent purchasing cost on pricing and production decisions, we next investigate two variants of the problem. In the first variant, we require that the firm utilizes the second chance opportunity of purchasing more supplies from other farmers whenever the realized supply is less than the demand. Because the purchasing alternative resembles backlogging, we term this as the complete backlogging variant. In the second variant, the firm is not allowed to make use of the purchasing alternative, representing the case of lost sales. From a managerial perspective, the optimal pricing and production choices differ in these two variants. From a technical perspective, the requirements in order to claim that the optimal policy is unique are also different for these two variants. Both variants prove to be useful in developing the solution approach for the problem described in this paper.

3.2 The Complete Backlogging variant

The case of complete backlogging corresponds to the scenario where the firm commits to purchasing the difference from other growers when the realized amount of supply cannot fulfill the demand. The optimal production decisions in stage 2 are $(q_1^*, q_2^*) = \left\{ \begin{array}{ll} (Qu, D(p) - Qu) & \text{when } Qu \leq D(p), \text{ and} \\ (D(p), 0) & \text{when } Qu > D(p) \end{array} \right\}$. Thus, q_2^* in the case of complete backlogging is equal to $D(p) - Qu$ whenever the realized supply is less than the demand, $Qu \leq D(p)$. Moreover, the total amount of production is always equal to the demand, i.e., $q_1^* + q_2^* = D(p)$. While the demand is guaranteed to be satisfied at all times in this variant of the problem, the ramifications of this commitment are such that there can be realizations of u where the unit purchasing cost exceeds the returns, i.e., $c_2(u) > p - c_p$, resulting in loss of money, rather than generating profits. We use the subscript CB in order to differentiate the decision variables and profit functions for the case of complete backlogging. Thus, the sale price and the stocking level are denoted as p_{CB} and z_{CB}, respectively. The first-stage objective function is then expressed as follows:

$$
\max_{(p_{CB}, z_{CB} \geq 0)} E\left[P_{E\_CB}(p_{CB}, z_{CB})\right] = \frac{D(p_{CB})}{z_{CB}} \left[\begin{array}{c} \left(p_{CB} - c_p - \frac{c}{\overline{u}}\right) z_{CB} \\ - \int_0^{z_{CB}} \left(\left(c_2(u) - \frac{c}{\overline{u}}\right)(z_{CB} - u)\right) g(u)\, du \\ - \left(\frac{c}{\overline{u}} - h_1\right) \int_{z_{CB}}^1 (u - z_{CB})\, g(u)\, du \end{array} \right] \qquad (25)
$$

where $E\left[P_{E\_CB}(p_{CB}, z_{CB})\right]$ is the expected profit under supply uncertainty with complete backlogging. In (25), $\left(p_{CB} - c_p - \frac{c}{\overline{u}}\right) D(p_{CB})$ corresponds to the profit from each unit of sale when supply is deterministic. The second term, $\frac{D(p_{CB})}{z_{CB}} \int_0^{z_{CB}} \left(\left(c_2(u) - \frac{c}{\overline{u}}\right)(z_{CB} - u)\right) g(u)\, du$ is the expected underage cost for the sale price p_{CB} and stocking level z_{CB}. For each unit of demand that was not filled through the growing option, $\left(c_2(u) - \frac{c}{\overline{u}}\right)$ is once again the incremental cost from purchasing the supply to satisfy the demand (rather than growing it). The third term $\frac{D(p_{CB})}{z_{CB}} \left(\frac{c}{\overline{u}} - h_1\right) \int_{z_{CB}}^1 (u - z_{CB})\, g(u)\, du$ is the expected overage cost for the sale price p_{CB} and stocking level z_{CB}. In the scenario where the realized supply exceeds the (expected) demand, each unit of excess supply costs $\frac{c}{u}$ to grow but brings back h_1 through salvaging.

The following theorem shows that the optimal stocking choice represented with variable z^*_{CB} can be determined independently of the sale price p_{CB}; thus, we can state that $z^*_{CB}(p_{CB}) = z^*_{CB}$ is a constant in the complete backlogging variant. Substituting the optimal value of the stocking level decision back into the first-stage objective function in (25) provides the optimal sale price defined as p^*_{CB}. The pair of optimal decisions (p^*_{CB}, z^*_{CB}) is then shown to be unique for this variant.

Theorem 2 *There exists a unique optimal solution for the first-stage objective function expressed in (25).*
a) The optimal stocking level decision z^*_{CB} in (25) can be determined independently of the sale price p_{CB}, and satisfies the following:

$$\int_0^{z^*_{CB}} \left[(c_2(u) - h_1)\, u\right] g(u)\, du = c - h_1 \overline{u}; \tag{26}$$

b) Substituting the optimal z^*_{CB} value obtained through (26) into (25) provides the optimal sale price p^*_{CB}, which satisfies the following:

$$p^*_{CB}\left(1 - \frac{1}{\in(p^*_{CB})}\right) = c_p + h_1 + \int_0^{z^*_{CB}} (c_2(u) - h_1)\, g(u)\, du. \tag{27}$$

When demand is linear in price as described in (1), the optimal sale price is:

$$p^*_{CB} = \frac{a + b\left(c_p + h_1 + \int_0^{z^*_{CB}} (c_2(u) - h_1)\, g(u)\, du\right)}{2b}, \tag{28}$$

and when demand is iso-elastic in price as described in (2), the optimal sale price is:

$$p^*_{CB} = \frac{b}{(b-1)}\left(c_p + h_1 + \int_0^{z^*_{CB}} (c_2(u) - h_1)\, g(u)\, du\right), \tag{29}$$

c) p^*_{CB} is increasing in z_{CB};
d) The optimal sale price under supply uncertainty for the complete backlogging variant is lower than or equal to that of the deterministic supply calculated on the basis of the expected unit purchasing cost; specifically, $p^*_{CB} \le p^{00}_p$;
e) When $c_2(\overline{u}) > E(c_2(u))$, the optimal sale price for the complete backlogging variant is less than or equal to that of the deterministic supply calculated on the basis of the unit purchasing cost at the expected yield; specifically, $p^*_{CB} \le p^{00}_p < p^0_p$;
f) The optimal sale price for the complete backlogging variant is greater than or equal to that of the deterministic supply calculated on the basis of the expected cost at the expected yield; specifically, $p^*_{CB} \ge p^0_g$.

It is necessary to further analyze the optimal sale price. The optimal sale price under the complete backlogging variant is less than or equal to its deterministic equivalent p^{00}_p calculated by using the expected unit purchasing cost $E(c_2(u))$, and is greater than or equal to the deterministic equivalent calculated on the basis of the expected cost of growing. However, a decisive conclusion cannot be made when the comparison involves the deterministic equivalent calculated on the basis of purchasing. When $c_2(\overline{u}) > E(c_2(u))$, for example, the optimal sale price is less than or equal to its deterministic equivalent, and $p^*_{CB} \le p^{00}_p < p^0_p$. However, when $c_2(\overline{u}) < E(c_2(u))$, the optimal sale price p_{CB}^* can be greater than its deterministic equivalent p_p^0 calculated by using the unit purchasing at $u = \overline{u}$, and the firm can have $p_p^0 \leq p_{CB}^* \leq p_p^{00}$. This is exemplified in the numerical illustrations that are presented in the Appendix (see Analysis 2 with convex cost).

The complete backlogging variant resembles the problem setting described in Li and Zheng (2006). Their paper also backlogs unsatisfied demand, however, this is accomplished at a static unit purchasing cost. Our model differs from their setting as the unit cost of the second chance opportunity for backlogging, $c_2(u)$, changes with the realized amount of supply. While their paper concludes that in the single-period problem the variability in supply uncertainty has no impact on the threshold of replenishment, it clearly influences the optimal stocking level in (26) and the optimal sale price in (27), (28) and (29). In order to compare the results in Theorem 2 with earlier research, consider the case where the unit purchasing cost is a constant and assume that the static unit purchasing cost is equal to $E(c_2(u))$. When $\int_0^1 c_2(u) u g(u) du > E(c_2(u)) \overline{u}$, z_{CB}^* is always smaller under the presence of the yield-dependent purchasing cost than that of the problem setting featuring a static cost. When this condition is not satisfied, however, the optimal stocking level can be lower or higher under the yield-dependent purchasing cost, depending on the problem parameters. Similarly, increasing variability in supply uncertainty does not always imply an increase in z_{CB}^* as expressed in (26); there can be cases where it reduces the value of z_{CB}^*. However, it can be easily observed that when the variability in supply uncertainty increases the stocking level z_{CB}^* in (26), it also increases the optimal sale price p_{CB}^*. As a result, the impact of variability in supply uncertainty has the same increasing (or decreasing) behavior on both the optimal stocking level and the sale price.

3.3 The Lost Sales variant

We next investigate the pricing and production decisions for the case where there is no second opportunity to obtain additional supplies. In this variant of the problem, q_2 is strictly equal to zero, and the firm cannot fulfill unsatisfied demand when the realized supply is lower than the (expected) demand. Therefore, the optimal production decision in stage 2 can be stated as $q_1^* = \left\{ \begin{array}{ll} Qu & \text{when } Qu \leq D(p) \text{, and} \\ D(p) & \text{when } Qu > D(p) \end{array} \right\}$. In this variant of the problem, when the realized supply is less than the demand, $Qu \leq D(p)$, the amount of unsatisfied demand equalling $D(p) - Qu$ corresponds to the amount of lost sales. We use the subscript LS in order to designate the lost sales variant, and the sale price and the stocking level are denoted by p_{LS} and z_{LS}, respectively. The first stage objective function is then expressed as follows:

$$\max_{(p_{LS}, z_{LS} \geq 0)} E\left[P_{E\_LS}(p_{LS}, z_{LS})\right] = \frac{D(p_{LS})}{z_{LS}} \left[\begin{array}{c} \left(p_{LS} - c_p - \frac{c}{\overline{u}}\right) z_{LS} \\ -\int_0^{z_{LS}} \left(\left(p_{LS} - c_p - \frac{c}{\overline{u}}\right)(z_{LS} - u)\right) g(u) du \\ -\left(\frac{c}{\overline{u}} - h_1\right) \int_{z_{LS}}^1 (u - z_{LS}) g(u) du \end{array} \right] \tag{30}$$

where $E\left[P_{E\_LS}(p_{LS}, z_{LS})\right]$ is the expected profit under supply uncertainty with lost sales. Once again, the term $\left(p_{LS} - c_p - \frac{c}{\overline{u}}\right) D(p_{LS})$ is the profit that the firm can obtain in the absence of supply uncertainty. For each unit of unsatisfied demand due to low supply realization, the firm pays a unit underage cost that is equal to the foregone profit of $\left(p_{LS} - c_p - \frac{c}{\overline{u}}\right)$. It should be noted that the unit underage cost for this variant of the problem is different than that of the complete backlogging variant, and is dependent on the choice of the sale price. For each unit of excess supply realization, the firm pays a unit overage cost of $\left(\frac{c}{\overline{u}} - h_1\right)$, which is equal to that of the complete backlogging variant. From underage cost, it can be seen that $p^*_{LS} > c_p + \frac{c}{\overline{u}}$.

Unlike the case of complete backlogging, the optimal stocking level z^*_{LS} cannot be determined independently of the sale price p_{LS}. Therefore, we follow a slightly different approach in order to obtain the structural results. Similar to the method employed by Zabel (1970) for the Price-Setting Newsvendor Problem (PSNP) under demand uncertainty, the objective function in (30) is first optimized for the sale price p_{LS}, and later solved for $z_{LS}(p_{LS})$ that maximizes $E\left[P_{E\_LS}(p_{LS}, z_{LS})\right]$. However, if the pdf $g(\cdot)$ for supply uncertainty is an arbitrary distribution, then it is necessary to conduct an exhaustive search over all values of z_{LS}. While no additional conditions on the pdf of supply uncertainty is required in the complete backlogging variant to prove the uniqueness of the optimal solution, it is necessary to enforce limitations in the lost sales variant. The following proposition shows that the solution proposed by this approach is the unique optimal solution for (30) when $g(\cdot)$ follows an increasing generalized failure rate (IGFR), which is defined as $r(z) = \frac{zg(z)}{1-G(z)}$. Distributions that follow the IGFR property are widely used in the literature, and a good discussion can be found in such articles as Barlow and Proschan (1965).

Proposition 3 *When the pdf of supply uncertainty with $g(\cdot)$ follows the IGFR property, there exists a unique optimal solution for the problem variant with lost sales with the objective function described in (30).*

a) The optimal stocking level $z^*_{LS}(p_{LS}) = 1$ for a given price level $p_{LS} \leq c_p + \frac{c}{\overline{u}}$, and can be obtained for a given price level $p_{LS} > c_p + \frac{c}{\overline{u}}$ as follows:

$$\int_0^{z^*_{LS}(p_{LS})} ug(u)\,du = \frac{c - h_1\overline{u}}{p_{LS} - c_p - h_1}; \tag{31}$$

b) $z^*_{LS}(p_{LS})$ is non-increasing in p_{LS};

c) The optimal sale price p^*_{LS} for a given z_{LS} satisfies:

$$p^*_{LS}(z_{LS})\left(1 - \frac{1}{\in\left(p^*_{LS}(z_{LS})\right)}\right) = c_p + h_1 + \frac{c - h_1\overline{u}}{\left[z_{LS} - \int_0^{z_{LS}}(z_{LS} - u)\,g(u)\,du\right]}. \tag{32}$$

When demand is linear in price as described in (1), the optimal sale price is:

$$p^*_{LS}(z_{LS}) = \frac{a + b\left(c_p + h_1 + \frac{c - h_1\overline{u}}{\left[z_{LS} - \int_0^{z_{LS}}(z_{LS}-u)g(u)du\right]}\right)}{2b}. \tag{33}$$

When demand is iso-elastic in price as described in (2), then the optimal sale price is:

$$p^*_{LS}(z_{LS}) = \frac{b}{(b-1)}\left(c_p + h_1 + \frac{c - h_1\overline{u}}{\left[z_{LS} - \int_0^{z_{LS}}(z_{LS} - u)\,g(u)\,du\right]}\right); \tag{34}$$

d) $p^*_{LS}(z_{LS})$ is non-increasing in z_{LS};

e) The optimal sale price under supply uncertainty for the lost sales variant is greater than or equal to that of the deterministic supply calculated on the basis of the expected unit cost of growing; specifically, $p^*_{LS}(z_{LS}) \geq p^0_g$.

Unlike the complete backlogging variant, the above proposition shows that the optimal stocking level in the lost sales variant $z^*_{LS}(p_{LS})$ is non-increasing in the sale price p_{LS}. Similarly, the optimal sale price is a function of the stocking level, however, its behavior with respect to z_{LS} is strikingly different from the complete backlogging variant. While the optimal sale price is increasing in the stocking level in the complete backlogging variant, the above proposition proves that $p_{LS}^*(z_{LS})$ is non-increasing in the stocking level. Once again, the optimal sale price under supply uncertainty p_{LS}^* is greater than or equal to its deterministic supply equivalent that is calculated on the basis of the expected cost of growing the fruit. Because the purchasing option is omitted in this variant of the problem, it is not possible to make a direct comparison with the deterministic supply equivalent that is based on the purchasing alternative. Now that the structural properties for the two problem variants with complete backlogging and lost sales are exhausted, we can turn our attention to the original problem that features both of the cases.

3.4 The Original Problem (EP)

The problem (EP) is a combination of the complete backlogging and lost sales variants, and incorporates their structural properties in its analysis. Our characterization of the optimal policy is based on the firm's behavior when its realized supply is not sufficient to fulfill the demand. The firm's optimal policy behavior can be described in three policies, and they are referred to as the Lost Sales (LS), Complete Backlogging (CB), and Combination (of lost sales and purchasing) policies. According to the LS policy, when the demand exceeds the realized supply, the firm does not exercise the purchasing option at any realization of u. In the CB policy, the firm utilizes the purchasing alternative at every realization of u, and acquires the difference between the demand and the realized supply no matter how expensive the yield-dependent unit purchasing cost becomes. The Combination policy can be perceived as partial lost sales and partial backlogging where the firm exercises the purchasing alternative based on the comparison of the sale price and the realized value of the yield parameter u. If the unit purchasing cost $c_2(u)$ is high enough such that the return $p - c_p$ does not justify the use of the purchasing alternative, the firm prefers to lose sales. However, if the unit purchasing cost is lower than $p - c_p$ at this value of u, then the firm utilizes the purchasing option. Thus, depending on the value of u and its corresponding value of $c_2(u)$, the firm prefers lost sales or purchasing in this policy.

The firm's use of the three potentially optimal policies can be presented in three regions that can be characterized as low, intermediate, and high price regions, and are denoted by subscripts I, II, and III, respectively. In the first region, the sale price is lower than the sum of the processing cost and the minimum yield-dependent purchasing cost, i.e., $p \leq c_p + c_2(u = 1)$. In this price region, the problem converges to the case of lost sales because the firm's optimal choice is to ignore the purchasing alternative at every realization of u even if there is unsatisfied demand. In the second region, the firm's sale price is less than the sum of processing and the maximum yield-dependent purchasing costs, but is greater than the sum of the processing and the minimum yield-dependent purchasing cost, i.e., $c_p + c_2(u = 1) < p < c_p + c_2(u = 0)$. One might intuit that the optimal policy behavior of the firm is always the Combination policy in this region. However, the optimal choice varies depending on the sale price in comparison to the yield-dependent purchasing cost. As will be shown later, the firm's optimal policy is not always the Combination policy in this region; it can be the LS or the Combination policy, but not the CB policy. In the third region when $p \geq c_p + c_2(u = 0)$, the problem converges to the complete backlogging variant, and the firm utilizes the purchasing option at every realization of u whenever the realized supply is less than the demand.

The analysis of (EP) proves that the optimal sale price can be in region II, and moreover, it can follow an LS policy. When this is the case, one would intuit that the firm's choice of an LS policy can occur only at lower prices such as the values that are closer to the sum of the processing and the minimum yield-dependent purchasing cost. However, as shown later, an LS policy can be optimal in intermediate values of the sale price in this region while having the Combination policy as the optimal choice at lower and higher price levels.

We begin our discussion with the optimal policy choice in region I. Note that when $p \leq c_p + c_2(u=1)$ the purchasing option is not desirable, and thus $q_2^* = 0$. The problem converges to the lost sales variant. It is important to observe that the stocking level for any price in region I is greater than that of the complete backlogging variant, and is equal to that of the lost sales variant. In this region, the expected profit from the lost sales variant of the problem is higher than that of the complete backlogging variant. The following proposition establishes the structural properties for the optimal policy.

Proposition 4 *For a given price p_{I} such that $p_{\mathrm{I}} \leq c_p + c_2(u=1)$, a) the stocking level is strictly greater than the optimal stocking level of the complete backlogging variant, i.e., $z_{\mathrm{I}}(p_{\mathrm{I}}) > z_{CB}^*$ for any price p_{I}; b) $E[P_E(p_{\mathrm{I}}, z_{CB}^*)] > E\left[P_{E\_CB}(p_{\mathrm{I}}, z_{CB}^*)\right]$ for any price p_{I}; c) If $p_{LS}^* \leq c_p + c_2(u=1)$, then $p_{\mathrm{I}}^* = p_{LS}^*$, $z_{\mathrm{I}}^* = z_{LS}^*$ and $E[P_E(p_{\mathrm{I}}^*, z_{\mathrm{I}}^*)] = E\left[P_{E\_LS}(p_{LS}^*, z_{LS}^*)\right]$; d) If $p_{LS}^* > c_p + c_2(u=1)$, then $z_{\mathrm{I}}^* = z_{LS}(p_{\mathrm{I}}^*) \geq z_{LS}^*$ and $E[P_E(p_{\mathrm{I}}^*, z_{\mathrm{I}}^*)] < E\left[P_{E\_LS}(p_{LS}^*, z_{LS}^*)\right]$, and as a result, the optimal solution for the problem (EP) can never be in region I. e) If $\frac{c}{\overline{u}} > c_p + c_2(u=1)$, then the optimal solution cannot be located in region I.*

The above proposition proves that the optimal solution for the problem (EP) can be in region I iff the optimal sale price for the lost sales variant is less than $c_p + c_2(u=1)$. However, this does not imply that the firm does not follow an LS policy in its optimal behavior. Indeed, as shown later, it can be the optimal choice in region II. The above proposition also shows that the stocking level in region I is greater than that of the complete backlogging variant.

We next present the analysis of region II where the sale price p_{II} is defined as $c_p + c_2(u=1) < p_{\mathrm{II}} < c_p + c_2(u=0)$. In this region, the firm's decision regarding whether to purchase more supplies from other growers depends on the sale price p_{II} and its comparison with the realized value of the unit purchasing cost $c_2(u)$ in addition to the unit pressing cost c_p. The firm engages in the purchasing alternative only when $p_{\mathrm{II}} > c_p + c_2(u)$. It is important to observe that it is not sufficient for the firm to set a sale price that is larger than $c_p + c_2(u=1)$ in order to justify the use of the second chance opportunity to purchase additional supplies. Indeed, setting the sale price greater than $c_p + c_2(u=1)$ does not eliminate the possibility of an LS policy from the set of optimal decisions. This observation necessitates a further analysis of the behavior of $c_2(u)$ and the yield parameter u that makes the purchasing option viable. Because the problem is a combination of the complete backlogging and lost sales variants, one would intuit that the firm's choice of the stocking level and sale price are intermediate points of these two variants. However, as shown in the following analysis, for a given price level p_{II}, the optimal stocking level $z_{\mathrm{II}}^*(p_{\mathrm{II}})$ is greater than or equal to those developed for the complete backlogging and lost sales variants.

Similarly, for a given stocking level z_{II}, the optimal sale price $p^{*}_{\mathrm{II}}(z_{\mathrm{II}})$ is greater than or equal to that of the lost sales variant.

The yield-dependent unit purchasing cost influences the firm's behavior significantly in this region. Let us define the inverse of the unit purchasing cost function as $c_2^{-1}(p_{\mathrm{II}} - c_p)$, where its value describes the minimum point for the realization of the yield parameter u to make the purchasing option a desirable alternative. More specifically, when the realized value of u is less than $c_2^{-1}(p_{\mathrm{II}} - c_p)$, the firm has $p_{\mathrm{II}} < c_p + c_2(u)$, and does not engage in the purchasing option, and when it is greater than $c_2^{-1}(p_{\mathrm{II}} - c_p)$, the firm utilizes the purchasing option. Note that when $p = c_p + c_2(u = 1)$, the minimum yield parameter is $c_2^{-1}(p - c_p) = 1$, and when $p = c_p + c_2(u = 0)$, it is equal to $c_2^{-1}(p - c_p) = 0$. Furthermore, the yield-dependent unit purchasing cost $c_2(u)$ is a decreasing function of u, and therefore its inverse function $c_2^{-1}(p_{\mathrm{II}} - c_p)$ is strictly decreasing from the value of 1 to the value of 0 as the sale price p_{II} increases from $c_p + c_2(u = 1)$ to $c_p + c_2(u = 0)$. It is shown earlier that a unique optimal solution is warranted under the IGFR condition for the lost sales variant of the problem. However, in order to ensure a unique optimal solution in region II of (EP), it is necessary to impose additional conditions. The new conditions establish the relationship between the yield-dependent purchasing cost function, the demand function, and the pdf of supply uncertainty. The following proposition provides a complete characterization of the optimal decisions.

Proposition 5 *In region II where the sale price is such that $c_p + c_2(u = 1) < p_{\mathrm{II}} < c_p + c_2(u = 0)$, if there exists a unique set of optimal stocking level and sale price choices, then it can be determined under the following properties of the yield-dependent purchasing cost function:*

$$\frac{-z_{\mathrm{II}}\left(\frac{\partial(c_2(z_{\mathrm{II}}))}{\partial z_{\mathrm{II}}}\right)}{(c_2(z_{\mathrm{II}}) - h_1)} < \frac{z_{\mathrm{II}}\left(\frac{\partial r(z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right)}{r(z_{\mathrm{II}})} \tag{35}$$

$$\frac{-p_{\mathrm{II}}\left(\frac{\partial c_2^{-1}(p_{\mathrm{II}}-c_p)}{\partial p_{\mathrm{II}}}\right)\left(z_{\mathrm{II}} - c_2^{-1}(p_{\mathrm{II}} - c_p)\right) g\left(c_2^{-1}(p_{\mathrm{II}} - c_p)\right)}{\left(z_{\mathrm{II}} - \int_0^{c_2^{-1}(p_{\mathrm{II}}-c_p)} (z_{\mathrm{II}} - u)\, g(u)\, du\right)} < 2 \in (p_{\mathrm{II}}) + \frac{p_{\mathrm{II}}\left(\frac{\partial^2 D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}^2}\right)}{\left(\frac{\partial D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}}\right)} \tag{36}$$

for a given z_{II} such that $z_{\mathrm{II}} > c_2^{-1}(p_{\mathrm{II}} - c_p)$.

a) When $\frac{c}{\overline{u}} > c_2(u = 1)$, there exists a minimum price level p_{II} denoted by $p^{\min}_{\mathrm{II}}$ which satisfies:

$$\left(p^{\min}_{\mathrm{II}} - c_p\right) \int_0^{c_2^{-1}\left(p^{\min}_{\mathrm{II}} - c_p\right)} u g(u)\, du + \int_{c_2^{-1}\left(p^{\min}_{\mathrm{II}} - c_p\right)}^{1} c_2(u)\, u g(u)\, du = c. \tag{37}$$

For a given sale price such that $c_p + c_2(u = 1) < p_{\mathrm{II}} \le p^{\min}_{\mathrm{II}}$, the optimal stocking level $z^{*}_{\mathrm{II}}(p_{\mathrm{II}}) = 1$. For a given sale price such that $p^{\min}_{\mathrm{II}} < p_{\mathrm{II}} < c_p + c_2(u = 0)$, the optimal stocking level can be obtained as follows:

$$\int_{c_2^{-1}(p_{\mathrm{II}}-c_p)}^{z^{*}_{\mathrm{II}}(p_{\mathrm{II}})} (c_2(u) - h_1)\, u g(u)\, du = c - h_1\overline{u} - (p_{\mathrm{II}} - c_p - h_1) \int_0^{c_2^{-1}(p_{\mathrm{II}}-c_p)} u g(u)\, du; \tag{38}$$

When $\frac{c}{\overline{u}} \le c_2(u = 1)$, the optimal stocking level can be obtained through (38) for all values of $c_p + c_2(u = 1) < p_{\mathrm{II}} < c_p + c_2(u = 0)$.

b) The optimal stocking level $z^{*}_{\mathrm{II}}(p_{\mathrm{II}})$ is non-increasing in sale price p_{II};

c) For a given stocking level z_{II}, when the yield-dependent purchasing cost function satisfies (36), there exists a *unique optimal sale price $p^*_{\text{II}}(z_{\text{II}})$ which satisfies the following:*

$$p^*_{\text{II}}(z_{\text{II}})\left(1-\frac{1}{\in\left(p^*_{\text{II}}(z_{\text{II}})\right)}\right)=c_p+h_1+\frac{\int_{c_2^{-1}(p_{\text{II}}-c_p)}^{z_{\text{II}}}(c_2(u)-h_1)(z_{\text{II}}-u)\,g(u)\,du}{\left(z_{\text{II}}-\int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(z_{\text{II}}-u)\,g(u)\,du\right)}+\frac{c-h_1\overline{u}}{\left(z_{\text{II}}-\int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(z_{\text{II}}-u)\,g(u)\,du\right)}. \tag{39}$$

When demand is linear in price as described in (1), the optimal sale price is:

$$p^*_{\text{II}}(z_{\text{II}})=\frac{a+b\left(c_p+h_1+\frac{\int_{c_2^{-1}(p_{\text{II}}-c_p)}^{z^*_{\text{II}}}(c_2(u)-h_1)(z_{\text{II}}-u)g(u)du}{\left(z_{\text{II}}-\int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(z_{\text{II}}-u)g(u)du\right)}+\frac{c-h_1\overline{u}}{\left(z_{\text{II}}-\int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(z_{\text{II}}-u)g(u)du\right)}\right)}{2b}. \tag{40}$$

When demand is iso-elastic in price as described in (2), then the optimal sale price is:

$$p^*_{\text{II}}(z_{\text{II}})=\frac{b}{(b-1)}\left(c_p+h_1+\frac{\int_{c_2^{-1}(p_{\text{II}}-c_p)}^{z^*_{\text{II}}}(c_2(u)-h_1)(z_{\text{II}}-u)\,g(u)\,du}{\left(z_{\text{II}}-\int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(z_{\text{II}}-u)\,g(u)\,du\right)}+\frac{c-h_1\overline{u}}{\left(z_{\text{II}}-\int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(z_{\text{II}}-u)\,g(u)\,du\right)}\right). \tag{41}$$

d) When the yield-dependent purchasing cost function satisfies (35) and (36), p^*_{II} and z^*_{II} provide the unique optimal solution for the problem (EP) in region II.

The above proposition provides the optimal decisions for the stocking level and the sale price. It shows that when the expected cost of growing the fruit is greater than the minimum cost of purchasing (i.e., $\frac{c}{\overline{u}}>c_2(u=1)$), the optimal sale price is guaranteed to be greater than a minimum established by $p^{\min}_{\text{II}}$ as expressed in (37). For price values that are less than or equal to $p^{\min}_{\text{II}}$, the optimal stocking level is equal to 1. It proves that the optimal stocking level can be determined uniquely for a given sale price; however, the sale price cannot be determined uniquely for a given stocking level without relying on (36). Note that (36) is satisfied when $z_{\text{II}}=c_2^{-1}(p_{\text{II}}-c_p)$, and is necessary only when $z_{\text{II}}>c_2^{-1}(p_{\text{II}}-c_p)$. Because the term $\left(\frac{\left(z_{\text{II}}-c_2^{-1}(p_{\text{II}}-c_p)\right)g\left(c_2^{-1}(p_{\text{II}}-c_p)\right)}{z_{\text{II}}-\int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(z_{\text{II}}-u)g(u)du}\right)$ is increasing in z_{II}, a sufficient condition can be developed by checking whether (36) is satisfied at $z_{\text{II}}=1$. In (36), the left-hand side is the price-elasticity of the term $\left(z_{\text{II}}-\int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(z_{\text{II}}-u)\,g(u)\,du\right)$. Because $c_2(u)$ is a decreasing function of u, $\frac{\partial c_2^{-1}(p_{\text{II}}-c_p)}{\partial p_{\text{II}}}<0$ and the price-elasticity of $\left(z_{\text{II}}-\int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(z_{\text{II}}-u)\,g(u)\,du\right)$ is positive. The right-hand side of (36) includes the price-elasticity of the demand function and a term that corresponds to the sale price times the percentage change in the slope of the demand function. The latter can be interpreted as the elasticity of the price elasticity of the demand function. When demand is linear as in (1), the right hand side of (36) is $\frac{2bp_{\text{II}}}{a-bp_{\text{II}}}$, and when it is iso-elastic as in (2), the right-hand-side (36) is $b-1>0$. Condition (35), on the other hand, establishes the relationship between the yield-dependent purchasing cost with the failure rate of the pdf of supply uncertainty. The left-hand side of (35) is the stocking-level elasticity of $(c_2(u)-h_1)$ corresponding to the firm's additional cost from the purchasing alternative. The right-hand side of (35) is the stocking-level elasticity of the failure rate $r(z_{\text{II}})$. Note that both sides of the inequality are positive, and (35) is a sufficient condition that warrants the uniqueness of the joint solution with p^*_{II} and z^*_{II}. In summary, (35) and (36) together provide restrictions on the shape of the yield-dependent purchasing cost relative to the demand function and the failure rate. It can be observed that when the unit purchasing cost is a constant and is not influenced by the yield (35) is satisfied but (36) is violated. When the inverse of the yield-dependent purchasing is a constant (and does not change with the yield), (36) is satisfied but (35) is violated. Thus, (35) and (36) together force $c_2(u)$ to decrease in yield and to be a well-behaving function.

The comparison of $c_2^{-1}(p_{\text{II}} - c_p)$ and $z_{LS}^*(p_{\text{II}})$ provides insight into the optimal policy behavior. It is necessary to highlight that these two functions can intersect many times, or not once, in region II depending on the form of $c_2(u)$ and $g(u)$. The following lemma establishes some useful properties between $c_2^{-1}(p_{\text{II}} - c_p)$ and $z_{LS}^*(p_{\text{II}})$.

Lemma 6 *a) At sale price $p = c_p + c_2(u = 0)$, $c_2^{-1}(p - c_p) = 0 < z_{LS}^*(p)$; b) When $c_2(u = 1) > \frac{c}{\bar{u}}$, at the sale price $p = c_p + c_2(u = 1)$ the firm has $c_2^{-1}(p - c_p) = 1 > z_{LS}^*(p)$; c) When $c_2(u = 1) < \frac{c}{\bar{u}}$, at the sale price $p = c_p + c_2(u = 1)$ the firm has $c_2^{-1}(p - c_p) = z_{LS}^*(p) = 1$; d) When $c_2(u = 1) < \frac{c}{\bar{u}}$, at the sale price $p = c_p + \frac{c}{\bar{u}}$ the firm has $c_2^{-1}(p - c_p) < z_{LS}^*(p) = 1$.*

When the demand function is described as in (1), the sale price is bounded from above with $\frac{a}{b}$ as demand becomes zero at values of $p \geq \frac{a}{b}$. Let us consider the case when the sum of the unit processing and the maximum yield-dependent purchasing cost, $c_p + c_2(u = 0)$, is less than the maximum price the firm can charge before demand reaches zero (this is not a concern for the iso-elastic demand function as described in (2)). The above lemma guarantees that $c_2^{-1}(p_{\text{II}} - c_p)$ and $z_{LS}^*(p_{\text{II}})$ intersect at least once when $c_2(u = 1) > \frac{c}{\bar{u}}$ and at least twice when $c_2(u = 1) \leq \frac{c}{\bar{u}}$. While $z_{LS}^*(p_{\text{II}})$ is decreasing in p_{II} as established in Proposition 3, $c_2^{-1}(p_{\text{II}} - c_p)$ is also decreasing in p_{II}. Moreover, the above lemma states that $c_2^{-1}(p_{\text{II}} - c_p)$ reaches zero at $p_{\text{II}} = c_p + c_2(u = 0)$, however, $z_{LS}^*(p_{\text{II}})$ is still positive at this price value. Because $c_2^{-1}(p_{\text{II}} - c_p)$ and $z_{LS}^*(p_{\text{II}})$ can intersect multiple times, it is beneficial to provide structural properties for the price intervals of p_{II} when $z_{LS}^*(p_{\text{II}}) > c_2^{-1}(p_{\text{II}} - c_p)$ and when $z_{LS}^*(p_{\text{II}}) \leq c_2^{-1}(p_{\text{II}} - c_p)$.

Proposition 7 *For a given price level p_{II}, a) when $z_{LS}^*(p_{\text{II}}) > c_2^{-1}(p_{\text{II}} - c_p)$, the optimal stocking level in region II is greater than or equal to the stocking levels obtained for the complete backlogging and lost sales variants. Specifically, i) $z_{\text{II}}^*(p_{\text{II}}) > z_{CB}^*(p_{\text{II}})$ and ii) $z_{\text{II}}^*(p_{\text{II}}) \geq z_{LS}^*(p_{\text{II}})$; b) when $z_{LS}^*(p_{\text{II}}) < c_2^{-1}(p_{\text{II}} - c_p)$, the optimal stocking level is equal to the lost sales variant and is greater than that of the complete backlogging variant, i.e., $z_{\text{II}}^*(p_{\text{II}}) = z_{LS}^*(p_{\text{II}}) > z_{CB}^*(p_{\text{II}})$.*

Proposition 8 *For a given stocking level z_{II}, a) when $z_{\text{II}} < c_2^{-1}(p_{LS}^*(z_{\text{II}}) - c_p)$, $p_{\text{II}}^*(z_{\text{II}}) = p_{LS}^*(z_{\text{II}})$; b) when $z_{\text{II}} > c_2^{-1}(p_{LS}^*(z_{\text{II}}) - c_p)$, the optimal sale price $p_{\text{II}}^*(z_{\text{II}})$ is greater than or equal to $p_{LS}^*(z_{\text{II}})$ and is less than or equal to $p_{CB}^*(z_{\text{II}})$, i.e., $p_{LS}^*(z_{\text{II}}) \leq p_{\text{II}}^*(z_{\text{II}}) \leq p_{CB}^*(z_{\text{II}})$; c) for any $z_{\text{II}} \geq z_{CB}^*$, $p_{LS}^*(z_{\text{II}}) \leq p_{CB}^*(z_{\text{II}})$.*

The consequence of Proposition 7 is that the stocking level of the complete backlogging variant serves as a lower bound to the problem in region II. Moreover, the optimal stocking level in this region is greater than or equal to the maximum of the two stocking levels developed for the lost sales and complete backlogging variants. It is already known from Proposition 3 that $p_{LS}^*(z_{\text{II}})$ is non-increasing in z_{II}. Combining the results of Proposition 7 and 8 for the values of $z_{\text{II}} \geq z_{CB}^*$, it can be seen that the optimal sale price for the lost sales variant is useful in determining the optimal choices in region II.

Theorem 9 *a) The optimal policy in region II can be an LS policy only when the optimal solution for the lost sales variant is such that* $z_{CB}^* < z_{LS}^* < c_2^{-1}(p_{LS}^*(z_{LS}^*) - c_p)$. *Otherwise, b) the optimal policy in region II is a Combination policy. When this is the case, the optimal stocking level is determined by (38) and its value is greater than or equal to those developed for the complete backlogging and lost sales variants, i.e.,* $z_{\mathrm{II}}^* > z_{LS}^*$ *and* $z_{\mathrm{II}}^* > z_{CB}^*$. *Moreover, the optimal sale price, as expressed in (39) is greater than or equal to that of the lost sales variant, but less than or equal to that of the complete backlogging variant, i.e.,* $p_{LS}^* \leq p_{\mathrm{II}}^* \leq p_{CB}^*$.

The above theorem highlights the significance of the yield-dependent purchasing cost in the optimal policy choice of the firm. First, it proves that for an LS policy to be optimal in region II, the optimal stocking level for the lost sales variant must be greater than its complete backlogging variant, warranting a lower sale price for the lost sales variant than the complete backlogging variant. In addition to this condition, the inverse of the yield-dependent purchasing cost at the optimal sale price of the lost sales variant must be greater than the optimal stocking level for the lost sales variant. In other words, at this stocking level, the purchasing option will not be economically viable by having $c_2(z_{LS}^*) > p_{LS}^*(z_{LS}^*) - c_p$. Only when these conditions are met, can an LS policy be a candidate for the optimal policy in region II; otherwise, the optimal policy is a Combination policy. Thus, the yield-dependent purchasing cost influences the optimal policy choice even in these intermediate values of the sale price. A second important result pertains to the relationship among stocking levels and prices. In the optimal choice, the firm prefers an optimal stocking level that is greater than or equal to that of those developed for both variants, and an optimal sale price that is greater than or equal to that of the lost sales variant but less than or equal to that of the complete backlogging variant. This is because while the CB policy drives price high to avoid a significant loss from the required purchasing cost, the LS policy has no pressure from the purchasing alternative to increase price.

In region III where $p \geq c_p + c_2(u = 0)$, the purchasing option is desirable for all realizations of u, and $q_2^* = D(p) - Qu$ whenever $Qu < D(p)$. The problem converges to the complete backlogging variant, and the stocking level in region III is equal to that of the complete backlogging variant, and is greater than that of the lost sales variant. The following proposition establishes the structural properties for the optimal policy.

Proposition 10 *For a given price* p_{III} *such that* $p_{\mathrm{III}} \geq c_p + c_2(u = 0)$, *a) the optimal stocking level is equal to that of the complete backlogging variant, i.e.,* $z_{\mathrm{III}}^*(p_{\mathrm{III}}) = z_{CB}^* > z_{LS}(p_{\mathrm{III}})$ *for any price* p_{III}; *b)* $E[P_E(p_{\mathrm{III}}, z_{\mathrm{III}}^*)] = E[P_{E\_CB}(p_{\mathrm{III}}, z_{CB}^*)] > E[P_{E\_LS}(p_{\mathrm{III}}, z_{\mathrm{III}}(p_{\mathrm{III}}))]$ *for any price* p_{III}; *c) If* $p_{CB}^* \geq c_p + c_2(u = 0)$, *then* $p_{\mathrm{III}}^* = p_{CB}^*$, $z_{\mathrm{III}}^* = z_{CB}^*$, *and* $E[P_E(p_{\mathrm{III}}^*, z_{\mathrm{III}}^*)] = E[P_{E\_CB}(p_{CB}^*, z_{CB}^*)]$; *d) If* $p_{CB}^* < c_p + c_2(u = 0)$, *then* $p_{\mathrm{III}}^* = c_p + c_2(u = 0)$, $z_{\mathrm{III}}^* = z_{CB}^*$, *however,* $E[P_E(p_{\mathrm{III}}^*, z_{\mathrm{III}}^*)] = E[P_{E\_CB}(p_{\mathrm{III}}^*, z_{CB}^*)] \leq E[P_{E\_CB}(p_{CB}^*, z_{CB}^*)]$; *and, as a result, the optimal solution cannot be located in region III.*

The above proposition provides several insights into the structural properties of the problem. First, the optimal solution for the problem (EP) can be in region III iff the optimal sale price for the complete backlogging variant is greater than or equal to $c_p + c_2(u = 0)$. From earlier analysis, recall that the optimal stocking level can be determined independently of the sale price in the complete backlogging variant. Furthermore, the objective function for the complete backlogging variant is concave in the sale price for a given value of the stocking level.

Therefore, if the optimal sale price is less than $c_p + c_2\,(u=0)$ the best choice of the optimal sale price turns out to be the threshold point of $p^*_{\text{III}} = c_p + c_2\,(u=0)$ with $z^*_{\text{III}} = z^*_{CB}$. Due to concavity, the objective function value for the optimal solution of the complete backlogging variant using a price lower than $c_p + c_2\,(u=0)$ presents a higher value than the best solution in region III. Thus, if the optimal sale price for the complete backlogging variant is less than $c_p + c_2\,(u=0)$, then a CB policy cannot be the optimal policy choice. It should be observed that the stocking level in this policy is equal to that of the complete backlogging variant, and is strictly greater than that of the lost sales variant. Combining the results from regions I, II and III, in all three potentially optimal policies, we can conclude that the optimal stocking level in (EP) is greater than or equal to those developed for the complete backlogging and lost sales variants.

The behavior of the optimal stocking level with respect to the sale price in all variants of the problem is exemplified in Figure 1. In problem (EP), $z^*(p)$ is decreasing in sale price p when $p < c_p + c_2\,(u=0)$, and becomes constant when the sale price exceeds the threshold point, i.e., $p \geq c_p + c_2\,(u=0)$. Moreover, in the sale price ranges where $z^*_{LS}(p_{\text{II}}) < c_2^{-1}(p_{\text{II}} - c_p)$, $z^*(p)$ behaves identically to the lost sales variant and its curve overlaps with $z^*_{LS}(p)$. In Figure 1, $z^*(p)$ and $z^*_{LS}(p)$ intersect with $c_2^{-1}(p - c_p)$ at the same two points. In price values that are lower than the smaller intersection point (in terms of price), $z^*(p)$ is decreasing in p steeper than $z^*_{LS}(p)$. This is because the price point that makes $z^*_{LS}(p) = 1$ is lower than that of $z^*(p) = 1$. After the second intersection point (corresponding to the larger price value), the decrease in $z^*(p)$ is not as steep as $z^*_{LS}(p)$ because there are values of u where the firm prefers to purchase supplies from other growers in order to fulfill unsatisfied demand. In this region of sale prices, the cost of acquiring additional supplies $c_2(u)$ is less than the foregone profit of $p - c_p$, resulting in firm's preference to increase the stocking level.

The following theorem provides the solution approach in order to obtain the optimal policy for the problem (EP). The solution approach compares the optimal solution obtained in each region. If the optimal sale price for the lost sales variant exceeds the threshold price point of $c_p + c_2\,(u=0)$, then the optimal decision is a CB policy. Otherwise, the comparison enables us to determine the optimal sale price and stocking level choices.

Theorem 11 *There are five possible cases that determine the optimal policy choices for the sale price and stocking level in problem* (EP)*.*

a) If $p^*_{LS} > c_p + c_2\,(u=1)$ *and* $p^*_{CB} < c_p + c_2\,(u=0)$*, then the optimal sale price is* $p^* = p^*_{\text{II}}$ *and the optimal stocking level is* $z^* = z^*_{\text{II}}(p^*_{\text{II}})$*. Note that the optimal choice in region II can be either an LS policy or a Combination Policy.*

b) If $p^*_{LS} \leq c_p + c_2\,(u=1)$ *and* $p^*_{CB} < c_p + c_2\,(u=0)$*, then the optimal policy can be obtained by comparing the objective function values of* $E[P_E(p^*_{LS}, z^*_{LS})]$ *with* $E[P_E(p^*_{\text{II}}, z^*_{\text{II}})]$*. If* $E[P_E(p^*_{LS}, z^*_{LS})] > E[P_E(p^*_{\text{II}}, z^*_{\text{II}})]$*, then the optimal sale price is* $p^* = p^*_{LS}$ *and the optimal stocking level is* $z^* = z^*_{LS}(p^*_{LS})$*; otherwise if* $E[P_E(p^*_{LS}, z^*_{LS})] \leq E[P_E(p^*_{\text{II}}, z^*_{\text{II}})]$*, then the optimal sale price is* $p^* = p^*_{\text{II}}$ *and the optimal stocking level is* $z^* = z^*_{\text{II}}(p^*_{\text{II}})$*.*

c) If $p^*_{LS} > c_p + c_2\,(u=1)$ *and* $p^*_{CB} \geq c_p + c_2\,(u=0)$*, then the optimal policy can be obtained by comparing the objective function values of* $E[P_E(p^*_{CB}, z^*_{CB})]$ *with* $E[P_E(p^*_{\text{II}}, z^*_{\text{II}})]$*. If* $E[P_E(p^*_{CB}, z^*_{CB})] > E[P_E(p^*_{\text{II}}, z^*_{\text{II}})]$*, then the optimal sale price is* $p^* = p^*_{CB}$ *and the optimal stocking level is* $z^* = z^*_{CB}$*; otherwise if* $E[P_E(p^*_{CB}, z^*_{CB})] \leq E[P_E(p^*_{\text{II}}, z^*_{\text{II}})]$*, then the optimal sale price is* $p^* = p^*_{\text{II}}$ *and the optimal stocking level is* $z^* = z^*_{\text{II}}(p^*_{\text{II}})$*.*

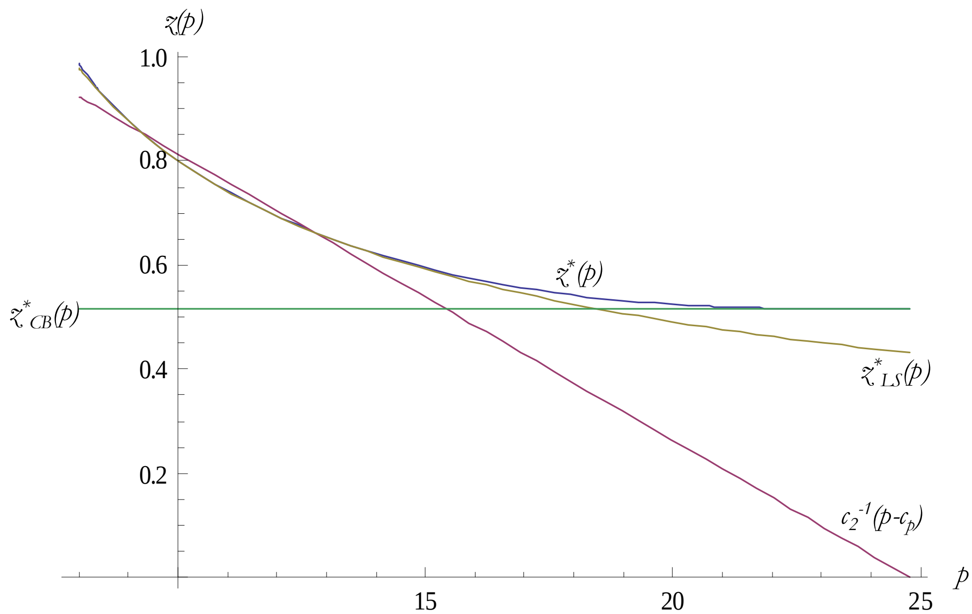

Figure 1: An example graph depicting how stocking level $z(p)$ changes with respect to sale price p.

d) If $p^*_{LS} \leq c_p + c_2\,(u=1)$ and $p^*_{CB} \geq c_p + c_2\,(u=0)$, then the optimal policy can be obtained by comparing the objective function values of $E\left[P_E\left(p^*_{LS}, z^*_{LS}\right)\right]$, $E\left[P_E\left(p^*_{\text{II}}, z^*_{\text{II}}\right)\right]$, and $E\left[P_E\left(p^*_{CB}, z^*_{CB}\right)\right]$. If $E\left[P_E\left(p^*_{LS}, z^*_{LS}\right)\right]$ has the highest value among three objective function values, then the optimal sale price is $p^* = p^*_{LS}$ and the optimal stocking level is $z^* = z^*_{LS}\left(p^*_{LS}\right)$; otherwise, if $E\left[P_E\left(p^*_{\text{II}}, z^*_{\text{II}}\right)\right]$ has the highest value among three objective function values, then the optimal sale price is $p^* = p^*_{\text{II}}$ and the optimal stocking level is $z^* = z^*_{\text{II}}\left(p^*_{\text{II}}\right)$; otherwise, if $E\left[P_E\left(p^*_{CB}, z^*_{CB}\right)\right]$ has the highest value among three objective function values, then the optimal sale price is $p^* = p^*_{CB}$ and the optimal stocking level is $z^* = z^*_{CB}$.
e) If $p^*_{LS} \geq c_p + c_2\,(u=0)$, the optimal policy is a CB policy, and the optimal sale price is $p^* = p^*_{CB}$ with the optimal stocking level $z^* = z^*_{CB}$.

There are important implications for the ordering of the sale prices and the comparison of the stocking levels. It has been proven that the optimal sale price is greater than or equal to its equivalent in the lost sales variant. This results in a lower demand than that of the lost sales variant; i.e., $D\left(p^*\right) \leq D\left(p^*_{LS}\right)$. It is also proven that the optimal stocking level is higher than its equivalent in the lost sales variant; i.e., $z^* \geq z^*_{LS}$. It can then can be concluded that the optimal amount of production targeted in the first stage (equivalently, the amount of farm space to be leased) is lower for the problem (EP) than that of the problem variant with lost sales.

Corollary 12 *The optimal amount of production in the first stage in (EP) is greater than or equal to that of the lost sales variant, i.e., $Q^* \leq Q^*_{LS}$.*

While the optimal stocking level is proven to be greater than or equal to that of the lost sales variant, the above theorem proves that the presence of a second-chance purchasing alternative reduces the original leasing investment Q only when the optimal solution is a Combination or a CB policy. A similarly decisive conclusion cannot be made when the problem (EP) is compared with the complete backlogging variant. It has been shown that the optimal sale price is lower than its equivalent in the complete backlogging variant; i.e., $p^* \leq p^*_{CB}$, and the optimal stocking level is higher than that of the complete backlogging variant; i.e., $z^* \geq z^*_{CB}$. However, it cannot be concluded that the optimal amount of production targeted in the first stage is always higher than its equivalent of the complete backlogging variant; $Q^* \geq Q^*_{CB}$. While this holds true generally, an example of a problem setting where $Q^* < Q^*_{CB}$ is provided in the numerical illustrations in the Appendix (for example, see Analysis 2 with convex cost).

It is the motivating argument of this paper that the yield-dependent unit purchasing cost provides a better representation of the operating environment for agricultural businesses and that it influences the firm's optimal decisions. The secondary chance of obtaining additional supplies is widely formulated with a static cost parameter in earlier research. The influence of the yield-dependent purchasing cost becomes clear in region II. If the unit purchasing cost were to be defined with a static cost parameter, the problem (EP) would not have featured region II, and the optimal solution would have been restricted to those presented in region I with an LS policy and in region III with a CB policy. The Combination policy is the outcome of the yield-dependent purchasing cost.

The following remark shows that the expected profit earned for the problem (EP) is greater than or equal to the objective function values obtained for the complete backlogging and lost sales variants.

Remark 13 *The expected profit obtained from the problem* (EP) *is greater than or equal to those developed for the complete backlogging and lost sales variants. Specifically,* $E\left[P_E\left(p^*, z^*\right)\right] \geq E\left[P_{E\_CB}\left(p^*_{CB}, z^*_{CB}\right)\right]$ *and* $E\left[P_E\left(p^*, z^*\right)\right] \geq E\left[P_{E\_LS}\left(p^*_{LS}, z^*_{LS}\right)\right]$.

Based on the above remark, the value gained from having the second chance opportunity of purchasing, denoted with V_E, can then be expressed as follows:

$$V_E = E\left[P_E\left(p^*, z^*\right)\right] - E\left[P_{E\_LS}\left(p^*_{LS}, z^*_{LS}\right)\right] \geq 0. \tag{42}$$

The purchasing alternative adds value to the firm when the optimal solution is either a CB or a Combination policy, and V_E becomes strictly positive. However, when the optimal choice is an LS policy, V_E becomes zero, and the purchasing alternative does not contribute to the firm's expected profit.

While the Early Pricing model provides a good benchmark for the problem, the Postponed Pricing model provides a better representation of the operating dynamics in the agricultural industry. Olive oil producers and orange juice producers, for example, set the sale price after they observe their crop, and therefore, benefit from the Postponed Pricing model. Next, we investigate the impact of the flexibility to postpone the pricing decision.

4 The Postponed Pricing Model

The Postponed Pricing model investigates the firm's ability to use pricing as a response mechanism to battle supply fluctuations. In this model, the firm decides on the optimal sale price after observing its realized supply. In the Early Pricing model, it is shown that the firm utilizes the purchasing alternative when the realized supply is less than the demand and when it is financially viable (i.e., when $p > c_p + c_2(u)$). In the Postponed Pricing model, denoted by (PP), the firm has two methods to deal with a low supply realization: 1) setting the sale price and influencing the demand, and 2) purchasing additional supplies from other growers at the yield-dependent purchasing cost. The investigation here presents whether it is necessary to use the purchasing option in the presence of pricing flexibility. If it is still viable to utilize the purchasing opportunity, then it is insightful to know under what conditions it is beneficial to exercise this option.

In the first stage of (PP), the firm determines the amount of farm space to be leased (Q), and pays cQ.

$$(PP): \max_{(Q \geq 0)} E[P_P(Q)] = -cQ + \int_0^1 PB(Q,u)\, g(u)\, du \tag{43}$$

where $PB(Q,u)$ is the recourse profit obtained from targeting the production of Q units (by leasing Q units of farm space), however, observing a yield of only Qu units.

In the second stage after observing the realized supply of Qu, the firm makes three decisions: 1) the sale price p, 2) the amount of final product obtained through an internally grown supply, which is denoted by q_1, and 3) the amount of final product produced from an externally purchased supply, which is denoted by q_2. Thus, the second-stage objective function can be written as follows:

$$\begin{aligned} PB(Q,u) &= \max_{(p,q_1,q_2 \geq 0)} \pi(p, q_1, q_2 \mid Q, u) = \left\{ \begin{array}{c} p \min\{(q_1+q_2), D(p)\} \\ -c_p(q_1+q_2) - c_2(u)\, q_2 + h_1 (Qu - q_1)^+ \end{array} \right\} \\ & \quad \text{s.t. } (5) \end{aligned} \tag{44}$$

It can be observed that given the cost structure in the problem, it is never beneficial for the firm to increase its total production beyond the demand; thus, $(q_1 + q_2)$ never exceeds $D(p)$, simplifying the objective function in (44). It is necessary to emphasize that the price-setting behavior in stage 2 is not a market-clearing strategy, but one that maximizes (44).

The optimal policy choices in the second stage of (PP) can be described in three classes based on the realized amount of supply. These three classes can be described as "low supply," "sufficient supply," and "excess supply." In the case of low supply, the firm prefers to utilize both pricing and external purchasing simultaneously. In the case of sufficient supply, the firm does not engage in the purchasing alternative, and sets the sale price to clear the realized supply. In the case of excess supply, the firm prefers not to further reduce the sale price to clear the realized supply. Instead, the price is kept at a certain level, and the excess of the demand is salvaged without being converted to the final product. These three optimal policy structures are summarized in the following proposition:

Proposition 14 *For a given realized supply of Qu, the optimal sale price p and production quantities q_1 and q_2 that maximize the objective function in (44) can be classified in three cases. When demand is linear in price as described in (1), the optimal decisions are as follows:*

$$(p^*, q_1^*, q_2^*) = \left\{ \begin{array}{ll} \left(\frac{a+b(c_p+c_2(u))}{2b}, Qu, \frac{a-b(c_p+c_2(u))}{2} - Qu\right) & \textit{when } Qu \leq \frac{a-b(c_p+c_2(u))}{2} \\ \left(\frac{a-Qu}{b}, Qu, 0\right) & \textit{when } \frac{a-b(c_p+c_2(u))}{2} < Qu \leq \frac{a-b(c_p+h_1)}{2} \\ \left(\frac{a+b(c_p+h_1)}{2b}, \frac{a-b(c_p+h_1)}{2}, 0\right) & \textit{when } Qu > \frac{a-b(c_p+h_1)}{2} \end{array} \right\}. \tag{45}$$

When demand is iso-elastic in price as described in (2), the optimal decisions are:

$$(p^*, q_1^*, q_2^*) = \left\{ \begin{array}{ll} \left(\frac{b}{b-1}\left(c_p + c_2\left(u\right)\right), Qu, a\left(\frac{b-1}{b(c_p+c_2(u))}\right)^b - Qu\right) & \text{when } Qu \le a\left(\frac{b-1}{b(c_p+c_2(u))}\right)^b \\ \left(\left(\frac{a}{Qu}\right)^{\frac{1}{b}}, Qu, 0\right) & \text{when } a\left(\frac{b-1}{b(c_p+c_2(u))}\right)^b < Qu \le a\left(\frac{b-1}{b(c_p+h_1)}\right)^b \\ \left(\frac{b}{b-1}\left(c_p + h_1\right), a\left(\frac{b-1}{b(c_p+h_1)}\right)^b, 0\right) & \text{when } Qu > a\left(\frac{b-1}{b(c_p+h_1)}\right)^b \end{array} \right\}. \tag{46}$$

The second stages of (EP) and (PP) feature three classes of potentially optimal policies, depending on the value of realized supply. While the two models show similarities, they differ characteristically on how and when they utilize the purchasing alternative. In (EP), when the firm experiences low supply realization, it does not purchase additional supplies (because the purchasing cost is higher than the return), whereas in (PP), the firm both adjusts the sale price and engages in external purchasing. When the realized supply is in intermediate values, the firm benefits from the purchasing opportunity in (EP); however, it prefers to adjust the price to clear the realized supply in (PP) without exercising the purchasing option. Both models lead to similar results in the case of excess supply, where the amount of production is not increased beyond a certain point.

It is necessary to further examine the behavior of the optimal values of the sale price and the total amount of production. Figure 2 depicts an example of how the optimal value of the sale price changes with respect to the initial investment Q in part (a) and with respect to u in part (b). The example considers a linearly decreasing unit purchasing cost, $c_2\left(u\right)$, and demand as in (1). For a given Q, the optimal value of the sale price is constant in the "low supply" region, is decreasing in the "sufficient supply" region, and is constant again in the "excess supply" region. However, for a given initial investment Q, as the yield parameter u increases, the optimal sale price in the "low supply" region decreases. This is because the yield-dependent unit purchasing cost becomes less expensive with increasing yield. It should be pointed out here that if the unit purchasing cost were to be defined as static, the optimal sale price would not have decreased and would have remained constant in this region. The optimal value of the sale price is decreasing in the "sufficient supply" region, and is constant in the "excess supply" region. Figure 3 shows an example graph of the optimal amount of production, $q_1^* + q_2^*$, with respect to the initial investment Q in part (a), and with respect to the yield parameter u in part (b). For a given Q, the optimal value of total production is constant in the "low supply" region, is increasing in the "sufficient supply" region, and is constant again in the "excess supply" region. However, for a given initial investment Q, as the realized yield increases, the optimal amount of total production increases in the "low supply" region. Once again, this is due to the fact that $c_2\left(u\right)$ becomes less expensive with increasing yield, and that the firm benefits more by taking advantage of this (less expensive) opportunity. If the unit purchasing cost were to be defined as static, the total production amount would not have increased but would have remained constant in this region. The optimal value of the sale price continues to increase in the "sufficient supply" region, and becomes constant in the "excess supply" region. As pointed out earlier, Figures 2 and 3 show that the yield-dependent purchasing cost impacts the optimal decisions regarding both the sale price and the production amount in the "low supply" region.

The recourse function described in (44) can be expressed as follows when (45) (or (46)) is substituted back

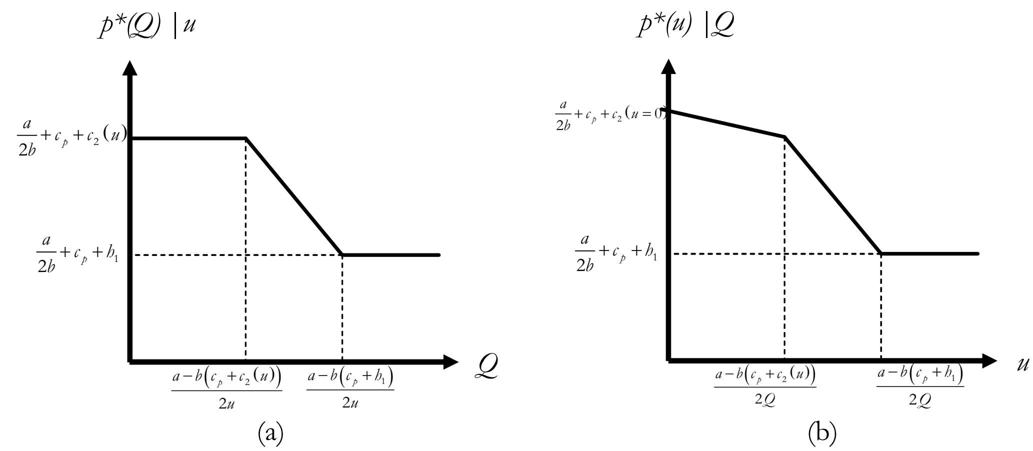

Figure 2: An example graph of the optimal sale price for a linearly decreasing yield-dependent unit purchasing cost. (a) shows how the sale price changes with respect to Q for a given value of u, and (b) shows how the sale price changes with respect to u for a given value of Q.

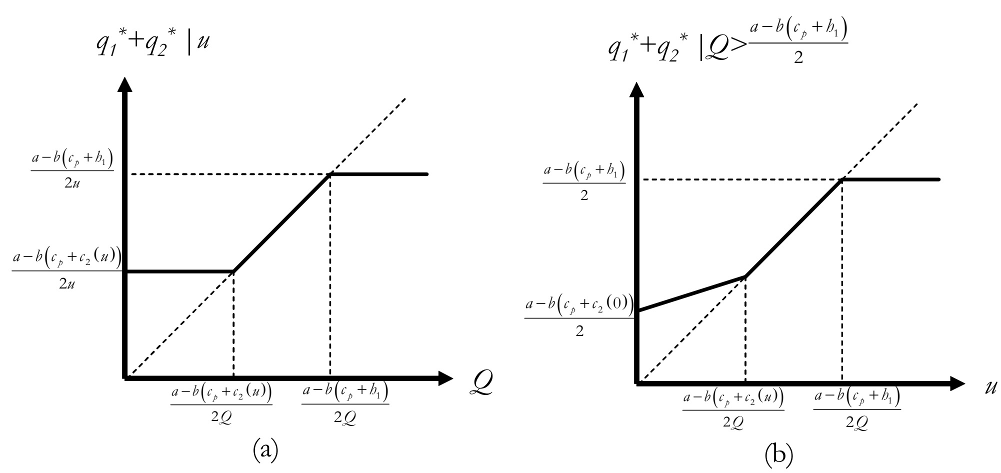

Figure 3: An example graph of the optimal amount of total production, $q_1^*+q_2^*$ with a linearly decreasing yield-dependent unit purchasing cost. (a) shows how the optimal amount of total production changes with respect to Q for a given u, and (b) shows how the optimal amount of total production changes with respect to u for a given Q.

into the second-stage objective function. When demand is linear in price as in (1), the recourse function is:

$$PB(Q,u) = \left\{ \begin{array}{ll} \frac{1}{b}\left(\frac{a-b(c_p+c_2(u))}{2}\right)^2 + c_2(u)\,Qu & \text{when } u \leq \frac{a-b(c_p+c_2(u))}{2Q} \\ \left(\frac{a-Qu}{b} - c_p\right)Qu & \text{when } \frac{a-b(c_p+c_2(u))}{2Q} < u \leq \frac{a-b(c_p+h_1)}{2Q} \\ \frac{1}{b}\left(\frac{a-b(c_p+h_1)}{2}\right)^2 + h_1 Qu & \text{when } u > \frac{a-b(c_p+h_1)}{2Q} \end{array} \right\}, \tag{47}$$

and when demand is iso-elastic in price as in (2), it is:

$$PB(Q,u) = \left\{ \begin{array}{ll} \frac{a}{b}\left(\frac{b-1}{b(c_p+c_2(u))}\right)^{b-1} + c_2(u)\,Qu & \text{when } u \leq \frac{a}{Q}\left(\frac{b-1}{b(c_p+c_2(u))}\right)^b \\ a^{\frac{1}{b}}(Qu)^{\frac{b-1}{b}} - c_p Qu & \text{when } \frac{a}{Q}\left(\frac{b-1}{b(c_p+c_2(u))}\right)^b < u \leq \frac{a}{Q}\left(\frac{b-1}{b(c_p+h_1)}\right)^b \\ \frac{a}{b}\left(\frac{b-1}{b(c_p+c_2(u))}\right)^{b-1} + h_1 Qu & \text{when } u > \frac{a}{Q}\left(\frac{b-1}{b(c_p+h_1)}\right)^b \end{array} \right\}. \tag{48}$$

Substituting the expressions in (47) and (48) into (43) provides the objective function value for the problem (PP). The following theorem states that the optimal decision regarding the amount of farm space to be leased can be uniquely determined because the objective function described in (43) is concave in Q. It should be mentioned that, unlike (EP), it is not necessary to impose any conditions on the pdf of supply uncertainty in (PP).

Theorem 15 *The objective function in (43) is continuous and concave in Q. Therefore, the optimal amount of Q^* can be uniquely determined.*

The value gained from the purchasing alternative can be calculated for the problem (PP). Representing this value with V_P, the purchasing alternative has a strictly positive impact on the firm's expected profit.

$$V_P = E\left[P_P(Q^*)\right] - E\left[P_P(Q^* \mid q_2^* = 0)\right] > 0. \tag{49}$$

When demand is linear as in (1),

$$V_P = \int_0^{\frac{a-b(c_p+c_2(u))}{2Q^*}} \left[\frac{1}{b}\left(\frac{a-b(c_p+c_2(u))}{2}\right)^2 - \left(\frac{a-Qu}{b} - c_p - c_2(u)\right)Q^*u\right] > 0,$$

and when demand is iso-elastic in price as in (2),

$$V_P = \int_0^{\frac{a}{Q^*}\left(\frac{b-1}{b(c_p+c_2(u))}\right)^b} \left[\frac{a}{b}\left(\frac{b-1}{b(c_p+c_2(u))}\right)^{b-1} - \left(a^{\frac{1}{b}}(Q^*u)^{\frac{b-1}{b}} - (c_p+c_2(u))\,Q^*u\right)\right] > 0.$$

Similarly, the firm gains value if it can postpone the pricing decisions until after the realization of the supply. The value gained from this action is denoted by V_{price} and can be expressed as:

$$V_{price} = E\left[P_P(Q^*)\right] - E\left[P_E(p^*, Q^*)\right] \geq 0.$$

5 Conclusions, Extensions, and Managerial Insights

This paper provides a review of the modeling approaches that investigate the impact of supply uncertainty on the interaction between operational and marketing-related decisions. The problem takes into account the availability of a second chance opportunity of purchasing additional supplies from other growers. While earlier studies typically formulate the unit purchasing cost as a static parameter, this paper presents a yield-dependent unit purchasing cost where the purchasing cost decreases with increasing yield; this definition provides a better representation of the working environment for agricultural business. A rich modeling framework is provided by 1) investigating the effects of setting prices before and after realizing supply uncertainty, 2) providing the solutions for two different demand functions (linearly decreasing and iso-elastic in price), and 3) investigating several variants of the problem with backlogging and lost sales, enabling us to establish the relationship with earlier research.

Supply uncertainty influences the optimal choices of the sale price and production quantities. In the Early Pricing model, two problem variants with complete backlogging and lost sales provide the foundation for the solution approach. In the case of complete backlogging where the firm is required to satisfy all demand, the optimal stocking level is a constant and can be determined independently of the sale price. The sale price increases with increasing values of the stocking level. When the firm cannot use the purchasing opportunity, as analyzed in the lost sales variant, the optimal stocking level decreases in the sale price, and the optimal sale price decreases in the stocking level. Incorporating the yield-dependent purchasing cost into the problem increases the complexity in developing a solution approach. The paper illustrates that it is not sufficient to set a sale price higher than the sum of the pressing cost and the minimum purchasing cost in order to satisfy the demand at all times. Instead, an optimal solution can feature a policy where the firm commits to lost sales despite having a price larger than the sum of the pressing and the minimum purchasing costs. The paper proves that the firm's optimal stocking level is always greater than or equal to those determined for the lost sales and the complete backlogging variants. The optimal sale price is greater than or equal to that of the lost sales variant, but is less than or equal to that of the complete backlogging variant. Furthermore, the optimal level of initial production (corresponding to the amount of farm space to be leased) for this problem is less than or equal to that of the lost sales variant. Therefore, as expected, the existence of a second chance opportunity to purchase additional supplies reduces the firm's original commitment to leasing farm land.

The influence of supply uncertainty can also be observed in the Postponed Pricing model. When the firm has the flexibility to postpone the pricing decision, from a technical perspective, the optimal amount of farm space to be leased can be determined easily. The sale price in this model decreases with the increasing values of the realized supply until a certain threshold is reached. It should be emphasized here that the decreasing behavior, in particular when the realized supply is low, is due to the presence of the yield-dependent purchasing cost. If the unit purchasing cost were to be defined as a static cost, the firm's optimal price choice would have been constant in low supply realizations.

Supply uncertainty is described by a stochastically proportional random variable equivalent to a multiplicative error term. The uncertainty in supply can also be defined with an additive random error term. As shown in Kazaz (2008), when supply uncertainty is described an additive random error term, the result regarding the optimal stocking levels remains the same in the Early Pricing model, and once again, its value is greater than or equal to those developed for the complete backlogging and lost sale variants. The optimal sale price is again greater than or equal to that of the lost sales variant, and is less than or equal to that of the complete backlogging variant. However, the optimal sale price is now less than or equal to the deterministic equivalent of the growing cost

. Thus, under the additive form of supply uncertainty, the firm shows an opposite reaction and reduces its sale price below its deterministic equivalent.

The analysis presented in this paper can be extended to problem settings that feature uncertain demand. In the absence of supply uncertainty, earlier research has shown that the optimal sale price increases when demand uncertainty has a multiplicative error term, and decreases with an additive random error term. In the Early Pricing model, when both sources of uncertainty are described by multiplicative error terms, the optimal sale price continues to increase beyond its deterministic equivalent; its value is greater than that of the deterministic demand and uncertain supply, and is also greater than that of the deterministic supply and uncertain demand. When both sources of uncertainty are described by additive error terms, the optimal sale price decreases beyond its deterministic equivalent; its value is less than that of the deterministic demand and uncertain supply, and is also less than that of deterministic supply and uncertain demand. When one source of uncertainty is defined with a multiplicative error term while the other with an additive error term, the two sources have opposite effects on the optimal sale price. Let us consider the case when the randomness in supply is described by a multiplicative error term and the randomness in demand is defined with an additive error term. Demand can be expressed as $D(p) + \varepsilon$ where ε is the additive random error term, $f(\varepsilon)$ represents its pdf and $F(\varepsilon)$ is the cdf. Under demand uncertainty, the firm may not be able to sell all of its production; it can salvage its unsold final product at a revenue of h_2 where it can be assumed that $h_2 < \left\{\frac{c}{u}, c_2(u=1)\right\}$ so that it does not increase its production in order to obtain salvage revenue. In the case of multiplicative supply error term and additive demand error term, supply uncertainty increases the optimal sale price while demand uncertainty decreases it. In the Early Pricing model, this paper has shown that the second stage decisions can be described in three regions of realized supply. The firm utilizes the purchasing alternative when the realized supply is less than the demand and when the yield-dependent purchasing cost is less than the sale price minus the processing cost. The latter leads to the result that the purchasing alternative is utilized when the yield is in between a minimum established by the inverse of the yield-dependent purchasing cost and a maximum established by the stocking level; specifically, when $c_2^{-1}(p - c_p) \leq u \leq z$. Under stochastic demand, the problem operates as if there are two suppliers. The first supplier has an uncertain output, and therefore, has a constrained capacity (equal to the amount of realized supply); thus, it can be perceived as a supplier who has a random capacity. In the absence of supply uncertainty, the firm would set its production amount to $D(p) + F^{-1}\left(\frac{p-c_p-h_1}{p-h_2}\right)$. The second supplier has an unlimited capacity and has no supply uncertainty, however, it can be perceived as operating with a random cost due to the yield-dependent purchasing cost. For a given value of the unit purchasing cost, the firm would set its production amount to $D(p) + F^{-1}\left(\frac{p-c_p-c_2(u)}{p-h_2}\right)$. Note that $F^{-1}\left(\frac{p-c_p-c_2(u)}{p-h_2}\right) < F^{-1}\left(\frac{p-c_p-h_1}{p-h_2}\right)$ for all u, resulting in a higher level of production coming from supplier 1 and a lower level of production from supplier 2. This observation leads to the result that the second-stage decisions can be described in four regions of realized supply (rather than three). The firm converts all of its realized supply when $Qu \leq D(p) + F^{-1}\left(\frac{p-c_p-h_1}{p-h_2}\right)$ and utilizes the purchasing alternative when $c_2^{-1}(p - c_p) \leq u \leq z\left(\frac{D(p)+F^{-1}\left(\frac{p-c_p-c_2(u)}{p-h_2}\right)}{D(p)+F^{-1}\left(\frac{p-c_p-h_1}{p-h_2}\right)}\right)$. While the same minimum is preserved, the maximum value of u that makes the purchasing alternative a viable option decreases below z under demand uncertainty because $\frac{D(p)+F^{-1}\left(\frac{p-c_p-c_2(u)}{p-h_2}\right)}{D(p)+F^{-1}\left(\frac{p-c_p-h_1}{p-h_2}\right)} < 1$. Thus, the purchasing alternative is beneficial in a smaller region of u values under demand uncertainty. The reduction in the range of u values that make the purchasing alternative viable is an outcome of the fact that the purchasing decisions are made before the uncertainty in demand is revealed. Its value would be higher if demand were to be realized before the purchasing decisions are made. Demand uncertainty can also be incorporated into the Postponed Pricing model, where the second stage corresponds to the Price-setting Newsvendor Problem (the characterization of this problem and the necessary optimality conditions are thoroughly presented in earlier research).

This paper assumes that the firm receives a constant salvage revenue in the event of excess supply. It can be argued that the salvage revenue can vary with the yield, requiring the firm to define it as yield-dependent. More specifically, the unit salvage revenue can also decrease with the increasing values of the yield parameter. When the model incorporates a yield-dependent salvage revenue, the results of the Early Pricing model get influenced, and a minimal effect can be observed in the Postponed Pricing model. In the lost sales variant, for example, at lower yield realizations, the salvage revenue can be higher than the sale price set in the first stage. This creates a range of yield parameter values where the firm prefers to sell its crop supply rather than converting it to the final product. This leads to an increased stocking level and initial production investment. Analytically, the lost sales variant of the problem now requires new conditions similar to those developed in (35) and (36) in addition to the IGFR property in order to warrant a unique optimal solution. It should be highlighted that the unit salvage revenue would be closely related with the unit purchasing cost. When the difference between the two functions is a constant, the impact of the yield-dependent salvage on pricing and production decisions is lessened in the original problem.

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Appendix A: Numerical Illustrations Using the Cost Parameters of a Turkish Olive Oil Producer

The examples presented in this numerical analysis provides are developed to illustrate the impact of the yield-dependent purchasing cost on the optimal sale price and production decisions under supply uncertainty. We are indebted to the Ayvalık Chamber of Commerce who provided the cost parameters and the estimates for the yield-dependent purchasing cost. In these examples, the cost parameters are stated in Turkish liras, and are equal to: $c = 2.43$, $c_p = 2.97$, $h_1 = 0.99$. The examples use a demand function that is linearly decreasing in price as described in (1), and is expressed as $D(p) = a - bp = 1,200,000 - 40,000p$. According to this demand function, the maximum value of the sale price can be 30 Turkish liras.

The numerical illustrations present the optimal decisions regarding the sale price and the stocking level under various forms of the yield-dependent purchasing cost function, denoted with $c_2(u)$. While the consensus among the managers of the olive oil producer and the members of the Ayvalık Chamber of Commerce is that the yield-dependent purchasing cost follows a convex or linear form in terms of the yield parameter u, the following analyses provides a comprehensive review with examples using convex, linear and concave relationships. As the Ayvalık Chamber of Commerce does not have sufficient data to provide an accurate description of $c_2(u)$, three different forms of calibration is used in order to estimate its functional form. In all three calibrations, the minimum value for the yield-dependent purchasing cost is used as $c_2(u = 1) = 3.64$, and a convex, a linear, and a concave function is developed using various criteria. In the first calibration, the analysis uses the yield-dependent purchasing cost at the mean value of the yield parameter where $c_2(\overline{u} = 0.5) = 12.73$, and forces each of the three functions (convex, linear, and concave) to pass through $c_2(u = 1) = 3.64$ and $c_2(\overline{u} = 0.5) = 12.73$. In Analysis 2, the maximum value of the yield-dependent purchasing cost is used for calibration, and each of the three functions are forced to pass through $c_2(u = 1) = 3.64$ and $c_2(u = 0) = 21.82$. In Analysis 3, the expected value of the yield-dependent purchasing cost is used for calibration, forcing all three functions to satisfy $c_2(u = 1) = 3.64$ and $E(c_2(u)) = 12.73$.

Before we present the analysis pertaining to supply uncertainty, let us determine the optimal sale price for the deterministic case:

$$p_g^0 = \frac{a + b\left(c_p + \frac{c}{\overline{u}}\right)}{2b} = 18.915.$$

The uncertainty in supply is described by a random error term which follows a uniform distribution where $u \in [0, 1]$ and the pdf is $g(u) = 1$.

It should be observed that the lost sales variant of the problem does not depend on the yield-dependent purchasing cost. Therefore, the optimal solution for this variant does not change with the examples developed later. The optimal solution for the lost sales variant is as follows:

$$z_{LS}^* = 0.49; p_{LS}^* = 19.567; E\left[P_{E\_LS}\left(p_{LS}^*, z_{LS}^*\right)\right] = 3,269,830; Q_{LS}^* = 838,054.$$

Analysis 1: Calibrating by $c_2(u = 1) = 3.64$ and $c_2(\overline{u} = 0.5) = 12.73$. Note that in the deterministic case using the yield-dependent purchasing cost at the expected value of the yield parameter, we have:

$$p_p^0 = \frac{a + b\left(c_p + c_2\left(\overline{u} = 0.5\right)\right)}{2b} = 22.85.$$

Convex Cost: $c_2(u) = 60.77259079 - 57.13259079u^{0.25}$

$$\begin{aligned} c_2(u=0) &= 60.77259079; c_2(u=1) = 3.64; E(c_2(u)) = 15.0665186 \\ p_p^0 &= 22.85; p_p^{00} = 24.018826 \\ z_{CB}^* &= 0.46; p_{CB}^* = 22.11; E\left[P_{E\_CB}(p_{CB}^*, z_{CB}^*)\right] = 2,489,930; Q_{CB}^* = 677,287 \\ z^* &= 0.52; p^* = 19.65; E\left[P_{E\_CB}(p^*, z^*)\right] = 3,283,430; Q^* = 796,154; c_2^{-1}(p^*) = 0.355 \end{aligned}$$

It can be easily observed that:

$$\begin{aligned} 1) z^* &> z_{LS}^* > z_{CB}^*, \\ 2) p_{LS}^* &< p^* < p_{CB}^*. \end{aligned}$$

Thus, the optimal policy is a Combination policy.

Linear Cost: Convex Cost: $c_2(u) = 21.82 - 18.18u$

$$\begin{aligned} c_2(u=0) &= 21.82; c_2(u=1) = 3.64; E(c_2(u)) = 12.73 \\ p_p^0 &= p_p^{00} = 22.85 \\ z_{CB}^* &= 0.515; p_{CB}^* = 21.14; E\left[P_{E\_CB}(p_{CB}^*, z_{CB}^*)\right] = 3,140,910; Q_{CB}^* = 688,162 \\ z^* &= 0.525; p^* = 19.83; E\left[P_{E\_CB}(p^*, z^*)\right] = 3,299,540; Q^* = 767,547; c_2^{-1}(p^*) = 0.273 \end{aligned}$$

It can be easily observed that:

$$\begin{aligned} 1) z^* &> z_{CB}^* > z_{LS}^*, \\ 2) p_{LS}^* &< p^* < p_{CB}^*. \end{aligned}$$

Thus, the optimal policy is a Combination policy.

Concave Cost: $c_2(u) = 13.336 - 9.696u^4$

$$\begin{aligned} c_2(u=0) &= 13.336; c_2(u=1) = 3.64; E(c_2(u)) = 11.3968 \\ p_p^0 &= 22.85; p_p^{00} = 22.1834 \\ z_{CB}^* &= 0.57; p_{CB}^* = 20.43; E\left[P_{E\_CB}(p_{CB}^*, z_{CB}^*)\right] = 3,665,750; Q_{CB}^* = 674,583 \\ z^* &= 0.57; p^* = 20.43; E\left[P_{E\_CB}(p^*, z^*)\right] = 3,665,750; Q^* = 674,583; c_2^{-1}(p^*) = 0 \end{aligned}$$

It can be easily observed that:

$$\begin{aligned} 1) z^* &= z_{CB}^* > z_{LS}^*, \\ 2) p_{LS}^* &< p^* = p_{CB}^*. \end{aligned}$$

Thus, the optimal policy is a CB policy.

Analysis 2: Calibrating by $c_2(u=1)=3.64$ and $c_2(u=0)=21.82$.

Convex Cost: $c_2(u)=21.82-18.18u^{0.25}$

$$\begin{aligned} c_2(u=0) &= 21.82; c_2(u=1)=3.64; E(c_2(u))=7.276; c_2(\overline{u}=0.5)=6.532503171 \\ p_p^0 &= 19.75125; p_p^{00}=20.123 \\ z_{CB}^* &= 0.85; p_{CB}^*=19.91; E\left[P_{E\_CB}(p_{CB}^*, z_{CB}^*)\right]=3,161,680; Q_{CB}^*=471,967 \\ z^* &= 0.86; p^*=19.88; E\left[P_{E\_CB}(p^*, z^*)\right]=4,077,704; Q^*=470,697; c_2^{-1}(p^*)=0.005. \end{aligned}$$

It can be easily observed that:

$$\begin{aligned} 1) z^* &> z_{CB}^* > z_{LS}^*, \\ 2) p_{LS}^* &< p^* < p_{CB}^*, \\ 3) p_p^0 &< p^* < p_{CB}^* < p_p^{00}, \\ 4) Q^* &< Q_{CB}^*. \end{aligned}$$

Thus, the optimal policy is a Combination policy.

Linear Cost: Convex Cost: $c_2(u)=21.82-18.18u$

$$\begin{aligned} c_2(u=0) &= 21.82; c_2(u=1)=3.64; E(c_2(u))=12.73 \\ p_p^0 &= p_p^{00}=22.85 \\ z_{CB}^* &= 0.515; p_{CB}^*=21.14; E\left[P_{E\_CB}(p_{CB}^*, z_{CB}^*)\right]=3,140,910; Q_{CB}^*=688,162 \\ z^* &= 0.525; p^*=19.83; E\left[P_{E\_CB}(p^*, z^*)\right]=3,299,540; Q^*=767,547; c_2^{-1}(p^*)=0.273. \end{aligned}$$

It can be easily observed that:

$$\begin{aligned} 1) z^* &> z_{CB}^* > z_{LS}^*, \\ 2) p_{LS}^* &< p^* < p_{CB}^*. \end{aligned}$$

Thus, the optimal policy is a Combination policy.

Concave Cost: $c_2(u)=21.82-18.18u^4$

$$\begin{aligned} c_2(u=0) &= 21.82; c_2(u=1)=3.64; E(c_2(u))=18.184 \\ p_p^0 &= 26.8266; p_p^{00}=25.577 \\ z_{CB}^* &= 0.43; p_{CB}^*=21.46; E\left[P_{E\_CB}(p_{CB}^*, z_{CB}^*)\right]=2,914,090; Q_{CB}^*=788,013 \\ z^* &= 0.49; p^*=19.57; E\left[P_{E\_CB}(p^*, z^*)\right]=3,269,830; Q^*=838,054; c_2^{-1}(p^*)=0.737. \end{aligned}$$

It can be easily observed that:

$$\begin{aligned} 1) z^* &= z_{LS}^* > z_{CB}^*, \\ 2) p_{LS}^* &< p^* = p_{CB}^*. \end{aligned}$$

Thus, the optimal policy is an LS policy.

Analysis 3: Calibrating by $c_2(u=1)=3.64$ and $E(c_2(u))=12.73$.

Convex Cost: $c_2(u)=49.09-45.45u^{0.25}$

$$\begin{aligned}
c_2(u=0) &= 49.09; c_2(u=1)=3.64; E(c_2(u))=12.73; c_2(\overline{u}=0.5)=11.27 \\
p_p^0 &= 22.12; p_p^{00}=22.85 \\
z_{CB}^* &= 0.534205; p_{CB}^*=21.52; E\left[P_{E_CB}(p_{CB}^*, z_{CB}^*)\right]=2,873,200; Q_{CB}^*=643,607 \\
z^* &= 0.56; p^*=19.82; E\left[P_{E_CB}(p^*, z^*)\right]=3,336,040; Q^*=727,142.9; c_2^{-1}(p^*)=0.253.
\end{aligned}$$

It can be easily observed that:

$$\begin{aligned}
1) z^* &> z_{CB}^* > z_{LS}^*, \\
2) p_{LS}^* &< p^* < p_{CB}^*.
\end{aligned}$$

Thus, the optimal policy is a Combination policy.

Linear Cost: Convex Cost: $c_2(u)=21.82-18.18u$

$$\begin{aligned}
c_2(u=0) &= 21.82; c_2(u=1)=3.64; E(c_2(u))=12.73 \\
p_p^0 &= p_p^{00}=22.85 \\
z_{CB}^* &= 0.515; p_{CB}^*=21.14; E\left[P_{E_CB}(p_{CB}^*, z_{CB}^*)\right]=3,140,910; Q_{CB}^*=688,162 \\
z^* &= 0.525; p^*=19.83; E\left[P_{E_CB}(p^*, z^*)\right]=3,299,540; Q^*=767,547; c_2^{-1}(p^*)=0.273.
\end{aligned}$$

It can be easily observed that:

$$\begin{aligned}
1) z^* &> z_{CB}^* > z_{LS}^*, \\
2) p_{LS}^* &< p^* < p_{CB}^*.
\end{aligned}$$

Thus, the optimal policy is a Combination policy.

Concave Cost: $c_2(u)=15.0025-11.3625u^4$

$$\begin{aligned}
c_2(u=0) &= 15.0025; c_2(u=1)=3.64; E(c_2(u))=14.29234375 \\
p_p^0 &= 23.63115; p_p^{00}=22.85 \\
z_{CB}^* &= 0.531282; p_{CB}^*=20.65; E\left[P_{E_CB}(p_{CB}^*, z_{CB}^*)\right]=3,494760; Q_{CB}^*=703,642 \\
z^* &= 0.531282; p^*=20.65; E\left[P_{E_CB}(p^*, z^*)\right]=3,494760; Q^*=703,642; c_2^{-1}(p^*)=0.
\end{aligned}$$

It can be easily observed that:

$$\begin{aligned}
1) z^* &= z_{CB}^* > z_{LS}^*, \\
2) p_{LS}^* &< p^* = p_{CB}^*.
\end{aligned}$$

Thus, the optimal policy is a CB policy.

Appendix B – Online Addendum

Proof of Theorem 2:

a) Consider the first-order derivative of the objective function in (25) with respect to z_{CB} for a given p_{CB}.

$$\begin{aligned}
\frac{\partial E\left[P_{E_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial z_{CB}} &= -\frac{D\left(p_{CB}\right)}{z_{CB}^2}\left[\begin{array}{c}\left(p_{CB}-c_p-\frac{c}{\overline{u}}\right) z_{CB} \\ -\int_0^{z_{CB}}\left(\left(c_2(u)-\frac{c}{\overline{u}}\right)\left(z_{CB}-u\right)\right) g(u)\, du \\ -\left(\frac{c}{\overline{u}}-h_1\right) \int_{z_{CB}}^1\left(u-z_{CB}\right) g(u)\, du\end{array}\right] \\
&\quad+\frac{D\left(p_{CB}\right)}{z_{CB}}\left[\left(p_{CB}-c_p-\frac{c}{\overline{u}}\right)-\int_0^{z_{CB}}\left(c_2(u)-\frac{c}{\overline{u}}\right) g(u)\, du+\left(\frac{c}{\overline{u}}-h_1\right) \int_{z_{CB}}^1 g(u)\, du\right] \\
&= \frac{D\left(p_{CB}\right)}{z_{CB}^2}\left[-\int_0^{z_{CB}}\left(\left(c_2(u)-h_1\right) u\right) g(u)\, du+c-h_1 \overline{u}\right] \\
&= \frac{D\left(p_{CB}\right)}{z_{CB}^2} R_z\left(z_{CB}\right) \text{ where} \\
R_z\left(z_{CB}\right) &= \left[-\int_0^{z_{CB}}\left(\left(c_2(u)-h_1\right) u\right) g(u)\, du+c-h_1 \overline{u}\right]
\end{aligned}$$

Note that when $z_{CB}=0$, $\frac{\partial E\left[P_{E\_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial z_{CB}}=\frac{D\left(p_{CB}\right)}{z_{CB}^2}\left[c-h_1 \overline{u}\right]>0$, and when $z_{CB}=1$, $\frac{\partial E\left[P_{E\_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial z_{CB}}=\frac{D\left(p_{CB}\right)}{z_{CB}^2}\left[-\int_0^1\left(c_2(u) u\right) g(u)\, du+c\right]<0$ because the expected cost of purchasing is more expensive than the cost of growing by definition. Moreover, $\frac{\partial R_z\left(z_{CB}\right)}{\partial z_{CB}}=-\left(\left(c_2\left(z_{CB}\right)-h_1\right) z_{CB}\right) g\left(z_{CB}\right)<0$, which implies that $R_z\left(z_{CB}\right)$ is decreasing in z_{CB} from a positive value at $z_{CB}=0$ to a negative value at $z_{CB}=1$. When $R_z\left(z_{CB}\right)$ reaches zero, we know it is a maximum because

$$\frac{\partial^2 E\left[P_{E\_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial z_{CB}^2} \Big|_{\frac{\partial E\left[P_{E\_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial z_{CB}}=0}=\frac{D\left(p_{CB}\right)}{z_{CB}^2}\left[-\left(\left(c_2\left(z_{CB}\right)-h_1\right) z_{CB}\right) g\left(z_{CB}\right)\right]<0.$$

Thus, (25) is concave in z_{CB} for a given p_{CB}, and the optimal value of z_{CB} can be determined independently of p_{CB}.

$$\begin{aligned}
\frac{\partial E\left[P_{E_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial p_{CB}} &= D\left(p_{CB}\right)+\left(\frac{\partial D\left(p_{CB}\right)}{\partial p_{CB}}\right) \frac{1}{z_{CB}}\left[\begin{array}{c}\left(p_{CB}-c_p-\frac{c}{\overline{u}}\right) z_{CB} \\ -\int_0^{z_{CB}}\left(\left(c_2(u)-\frac{c}{\overline{u}}\right)\left(z_{CB}-u\right)\right) g(u)\, du \\ -\left(\frac{c}{\overline{u}}-h_1\right) \int_{z_{CB}}^1\left(u-z_{CB}\right) g(u)\, du\end{array}\right] \\
\frac{\partial^2 E\left[P_{E_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial p_{CB}^2} &= 2\left(\frac{\partial D\left(p_{CB}\right)}{\partial p_{CB}}\right)+\left(\frac{\partial^2 D\left(p_{CB}\right)}{\partial p_{CB}^2}\right) \frac{1}{z_{CB}}\left[\begin{array}{c}\left(p_{CB}-c_p-\frac{c}{\overline{u}}\right) z_{CB} \\ -\int_0^{z_{CB}}\left(\left(c_2(u)-\frac{c}{\overline{u}}\right)\left(z_{CB}-u\right)\right) g(u)\, du \\ -\left(\frac{c}{\overline{u}}-h_1\right) \int_{z_{CB}}^1\left(u-z_{CB}\right) g(u)\, du\end{array}\right] \\
\frac{\partial^2 E\left[P_{E_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial p_{CB}^2} &\Big|_{\frac{\partial E\left[P_{E_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial p_{CB}}=0}=\frac{1}{\left(\frac{\partial D\left(p_{CB}\right)}{\partial p_{CB}}\right)}\left[2\left(\frac{\partial D\left(p_{CB}\right)}{\partial p_{CB}}\right)^2+\left(\frac{\partial^2 D\left(p_{CB}\right)}{\partial p_{CB}^2}\right) D\left(p_{CB}\right)\right]
\end{aligned}$$

When demand is linear in price as defined in (1), $\frac{\partial D\left(p_{CB}\right)}{\partial p_{CB}}=-b<0$ and $\frac{\partial^2 D\left(p_{CB}\right)}{\partial p_{CB}^2}=0$. Therefore, $\frac{\partial^2 E\left[P_E\left(p_{CB}, z_{CB}\right)\right]}{\partial p_{CB}^2}=-2 b<0$, and (25) is concave in p_{CB} for a given z_{CB}. Similarly, when demand is iso-elastic as in equation (2), $\frac{\partial D\left(p_{CB}\right)}{\partial p_{CB}}=-\frac{b}{p_{CB}} D\left(p_{CB}\right)<0$ and $\frac{\partial^2 D\left(p_{CB}\right)}{\partial p_{CB}^2}=\frac{b(b+1)}{p_{CB}^2} D\left(p_{CB}\right)>0$. Because $\left(\frac{\partial D\left(p_{CB}\right)}{\partial p_{CB}}\right)^2>0$ and $\left(\frac{\partial^2 D\left(p_{CB}\right)}{\partial p_{CB}^2}\right) D\left(p_{CB}\right)>0$, we have $\left[2\left(\frac{\partial D\left(p_{CB}\right)}{\partial p_{CB}}\right)^2+\left(\frac{\partial^2 D\left(p_{CB}\right)}{\partial p_{CB}^2}\right) D\left(p_{CB}\right)\right]>0$. Since $\frac{D\left(p_{CB}\right)}{\partial p_{CB}}<0$ we get $\frac{\partial^2 E\left[P_{E\_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial p_{CB}^2} \Big|_{\frac{\partial E\left[P_{E\_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial p_{CB}}=0}<0$; thus (25) is concave in p_{CB} for a given z_{CB}. The cross-derivative of (25) is:

$$\frac{\partial^2 E\left[P_{E\_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial p_{CB}\partial z_{CB}}\mid_{\frac{\partial E\left[P_{E\_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial z_{CB}}=0} = \left(\frac{\partial D\left(p_{CB}\right)}{\partial p_{CB}}\right)\frac{1}{z_{CB}^2}\left[-\int_0^{z_{CB}}\left(\left(c_2\left(u\right)-h_1\right)u\right)g\left(u\right)du + c - h_1\overline{u}\right] = 0.$$

As a result, the determinant of the Hessian is positive for both the linear and iso-elastic demand functions as expressed in (1) and (2). Therefore, the optimal z_{CB} value that satisfies $\frac{\partial E\left[P_{E\_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial z_{CB}} = 0$ is a unique maximizer.

b) From $\frac{\partial E\left[P_{E\_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial z_{CB}} = 0$, it is known that $\int_0^{z_{CB}}\left(\left(c_2\left(u\right)-h_1\right)u\right)g\left(u\right)du = c - h_1\overline{u}$, and substituting this expression into $\frac{\partial E\left[P_{E\_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial p_{CB}} = 0$ using the linear form of demand provides the following:

$$p_{CB}^* = \frac{a + b\left(c_p + h_1\right) + b\int_0^{z_{CB}^*}\left(c_2\left(u\right)-h_1\right)g\left(u\right)du}{2b}.$$

Similarly, substituting $\int_0^{z_{CB}}\left(\left(c_2\left(u\right)-h_1\right)u\right)g\left(u\right)du = c - h_1\overline{u}$ into $\frac{\partial E\left[P_{E\_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial p_{CB}} = 0$ using the iso-elastic form of demand provides:

$$p_{CB}^* = \frac{b}{(b-1)}\left(c_p + h_1\right) + \frac{b}{(b-1)}\int_0^{z_{CB}^*}\left(c_2\left(u\right)-h_1\right)g\left(u\right)du.$$

Note that $\frac{\partial E\left[P_{E\_CB}\left(p_{CB}, z_{CB}\right)\right]}{\partial p_{CB}} = 0$ corresponds to:

$$\begin{aligned}
D\left(p_{CB}\right) + \left(\frac{\partial D\left(p_{CB}\right)}{\partial p_{CB}}\right)\frac{1}{z_{CB}}\left[\begin{array}{c}\left(p_{CB} - c_p - \frac{c}{\overline{u}}\right)z_{CB} \\ -\int_0^{z_{CB}}\left(\left(c_2\left(u\right) - \frac{c}{\overline{u}}\right)\left(z_{CB} - u\right)\right)g\left(u\right)du \\ -\left(\frac{c}{\overline{u}} - h_1\right)\int_{z_{CB}}^1\left(u - z_{CB}\right)g\left(u\right)du\end{array}\right] &= 0 \\
\left[\begin{array}{c}\left(p_{CB} - c_p - h_1\right)z_{CB} - \left(c - h_1\overline{u}\right) \\ -\int_0^{z_{CB}}\left(\left(c_2\left(u\right) - h_1\right)\left(z_{CB} - u\right)\right)g\left(u\right)du\end{array}\right] &= -\frac{z_{CB}D\left(p_{CB}\right)}{\left(\frac{\partial D\left(p_{CB}\right)}{\partial p_{CB}}\right)}
\end{aligned}$$

$$\left(p_{CB} - c_p - h_1\right)z_{CB} = \left(c - h_1\overline{u}\right) + \int_0^{z_{CB}}\left[\left(\left(c_2\left(u\right) - h_1\right)\left(z_{CB} - u\right)\right)g\left(u\right)du\right] - \frac{z_{CB}D\left(p_{CB}\right)}{\left(\frac{\partial D\left(p_{CB}\right)}{\partial p_{CB}}\right)}$$

$$p_{CB}^* = c_p + h_1 + \int_0^{z_{CB}^*}\left(c_2\left(u\right)-h_1\right)g\left(u\right)du - \frac{D\left(p_{CB}^*\right)}{\left(\frac{\partial D\left(p_{CB}\right)}{\partial p_{CB}}\mid_{p_{CB}=p_{CB}^*}\right)}$$

Expressing the price elasticity of demand with $\in(p) = \frac{-p\left(\frac{\partial D\left(p_{CB}\right)}{\partial p_{CB}}\right)}{D(p)}$, the optimal sale price can be written as:

$$p_{CB}^*\left(1 - \frac{1}{\in\left(p_{CB}^*\right)}\right) = c_p + h_1 + \int_0^{z_{CB}^*}\left(c_2\left(u\right)-h_1\right)g\left(u\right)du$$

c) Because $\frac{\partial p_{CB}^*}{\partial z_{CB}} = \left(c_2\left(z_{CB}\right) - h_1\right)g\left(z_{CB}\right) > 0$, the optimal sale price p_{CB}^* is increasing in z_{CB}.

d) From (27), we have

$$\begin{aligned}
p_{CB}^* &= c_p + h_1 + \int_0^{z_{CB}^*}\left(c_2\left(u\right)-h_1\right)g\left(u\right)du - \frac{D\left(p_{CB}^*\right)}{\left(\frac{\partial D\left(p_{CB}\right)}{\partial p_{CB}}\mid_{p_{CB}=p_{CB}^*}\right)} \\
&\le c_p + h_1 + \int_0^1\left(c_2\left(u\right)-h_1\right)g\left(u\right)du - \frac{D\left(p_{CB}^*\right)}{\left(\frac{\partial D\left(p_{CB}\right)}{\partial p_{CB}}\mid_{p_{CB}=p_{CB}^*}\right)} = c_p + E\left(c_2\left(u\right)\right) - \frac{D\left(p_{CB}^*\right)}{\left(\frac{\partial D\left(p_{CB}\right)}{\partial p_{CB}}\mid_{p_{CB}=p_{CB}^*}\right)} \\
p_{CB}^* &\le p_p^{00}.
\end{aligned}$$

e) When $c_2(\overline{u}) > E(c_2(u))$, it can be seen from the analysis of the deterministic supply that $p_p^{00} < p_p^0$. From part d), it is known that $p_{CB}^* \leq p_p^{00}$, therefore, $p_{CB}^* \leq p_p^{00} < p_p^0$.

f) In the analysis of the deterministic analysis based on the expected unit cost of growing, the firm has $p_g^0 = c_p + \frac{c}{\overline{u}} - \left(\frac{D(p)}{\frac{\partial D(p)}{\partial p}}\right)$. From (26) it is known that $h_1 = \frac{c}{\overline{u}} - \frac{1}{\overline{u}} \int_0^{z_{CB}^*} [(c_2(u) - h_1)u]\, g(u)\, du$; and substituting the expression for h_1 into (27) provides the following:

$$
\begin{aligned}
p_{CB}^* &= c_p + \left(\frac{c}{\overline{u}} - \frac{1}{\overline{u}} \int_0^{z_{CB}^*} [(c_2(u) - h_1)u]\, g(u)\, du\right) + \int_0^{z_{CB}^*} (c_2(u) - h_1)\, g(u)\, du - \left(\frac{D(p)}{\left(\frac{\partial D(p)}{\partial p}\right)}\right) \\
p_{CB}^* &= c_p + \frac{c}{\overline{u}} + \int_0^{z_{CB}^*} \left((c_2(u) - h_1)\left(1 - \frac{u}{\overline{u}}\right)\right) g(u)\, du - \left(\frac{D(p)}{\left(\frac{\partial D(p)}{\partial p}\right)}\right) \geq p_g^0
\end{aligned}
$$

because $c_2(u)$ is a decreasing function, and while $\int_0^{z_{CB}^*} \left((c_2(u) - h_1)\left(1 - \frac{u}{\overline{u}}\right)\right) g(u)\, du$ is decreasing as the value of z_{CB}^* increases, its value is always positive even at $z = 1$ (i.e., the term $\int_0^1 \left((c_2(u) - h_1)\left(1 - \frac{u}{\overline{u}}\right)\right) g(u)\, du > 0$).

Proof of Proposition 3:

a) The objective function in (30) is equivalent to the following:

$$
E\left[P_{E\_LS}(p_{LS}, z_{LS})\right] = \frac{D(p_{LS})}{z_{LS}} \left[(p_{LS} - c_p - h_1)\left(z_{LS} - \int_0^{z_{LS}} (z_{LS} - u)\, g(u)\, du\right) - (c - h_1\overline{u})\right]
$$

The first-order derivative of the above objective function with respect to z_{LS} is:

$$
\begin{aligned}
\frac{\partial E\left[P_{E\_LS}(p_{LS}, z_{LS})\right]}{\partial z_{LS}} &= \frac{D(p_{LS})}{z_{LS}^2} \begin{bmatrix} -(p_{LS} - c_p - h_1)\left(z_{LS} - \int_0^{z_{LS}} (z_{LS} - u)\, g(u)\, du\right) + (c - h_1\overline{u}) \\ +(p_{LS} - c_p - h_1)\, z_{LS}\,(1 - G(z_{LS})) \end{bmatrix} \\
&= \frac{D(p_{LS})}{z_{LS}^2} R_z(p_{LS}, z_{LS}) \text{ where} \\
R_z(p_{LS}, z_{LS}) &= \left[-(p_{LS} - c_p - h_1)\int_0^{z_{LS}} u g(u)\, du + c - h_1\overline{u}\right]
\end{aligned}
$$

Note that when $z_{LS} = 0$, $R_z(p_{LS}, z_{LS}) = [c - h_1\overline{u}]$ and $\frac{\partial E\left[P_{E\_LS}(p_{LS}, z_{LS})\right]}{\partial z_{LS}} = \frac{D(p_{LS})}{z_{LS}^2}[c - h_1\overline{u}] > 0$. When $z_{LS} = 1$, we have $R_z(p_{LS}, z_{LS}) = [-(p_{LS} - c_p)\overline{u} + c]$. For price values $p_{LS} \leq c_p + \frac{c}{\overline{u}}$, we have $R_z(p_{LS}, z_{LS}) = [-(p_{LS} - c_p)\overline{u} + c] > 0$, and $\frac{\partial E\left[P_{E\_LS}(p_{LS}, z_{LS})\right]}{\partial z_{LS}} > 0$. Thus, the objective function continues to improve even at $z_{LS} = 1$. It should also be observed that

$$
\frac{\partial E\left[P_{E\_LS}(p_{LS}, z_{LS} = 1)\right]}{\partial p_{LS}} = \left(\frac{\partial D(p_{LS})}{\partial p_{LS}}\right)(\overline{u})\left[\frac{D(p_{LS})}{\left(\frac{\partial D(p_{LS})}{\partial p_{LS}}\right)} + \left(p_{LS} - c_p - \frac{c}{\overline{u}}\right)\right] > 0
$$

in the price range $p_{LS} \leq c_p + \frac{c}{\overline{u}}$, therefore, $z_{LS}^* = 1$ in the price range $p_{LS} \leq c_p + \frac{c}{\overline{u}}$. Moreover, $p_{LS} = c_p + \frac{c}{\overline{u}}$ serves as the minimum price range where $z_{LS}^* > 1$, and as will be shown later, the value of $z_{LS}(p_{LS})$ decreases for price values $p_{LS} > c_p + \frac{c}{\overline{u}}$. Observe that $\frac{\partial R_z(p_{LS}, z_{LS})}{\partial z_{LS}} = -(p_{LS} - c_p - h_1)\, z_{LS}\, g(z_{LS}) < 0$ for price values $p_{LS} > c_p + \frac{c}{\overline{u}}$, and $R_z(p_{LS}, z_{LS})$ decreases in z_{LS} from a positive value at $z_{LS} = 0$ to a negative value at $z_{LS} = 1$. Thus, the value of $\frac{\partial E\left[P_{E\_LS}(p_{LS}, z_{LS})\right]}{\partial z_{LS}}$ switches from a positive value to a negative value as z_{LS} increases from 0 to 1 in the price range $p_{LS} > c_p + \frac{c}{\overline{u}}$. Thus, equating $\frac{\partial E\left[P_{E\_LS}(p_{LS}, z_{LS})\right]}{\partial z_{LS}} = 0$ provides the optimal stocking level z_{LS}^* in equation (32) for a given sale price p_{LS}:

$$
\int_0^{z_{LS}^*} u g(u)\, du = \frac{c - h_1\overline{u}}{p_{LS} - c_p - h_1}.
$$

The second-order derivative with respect to z_{LS} at the optimal stocking point shows that (31) is a maximizer for a given p_{LS}, because

$$\begin{aligned}\frac{\partial^2 E\left[P_{E\_LS}\left(p_{LS}, z_{LS}\right)\right]}{\partial z_{LS}^2} &= \frac{-2D\left(p_{LS}\right)}{z_{LS}^2}\left[-\left(p_{LS}-c_p-h_1\right)\int_0^{z_{LS}} ug\left(u\right)du + c - h_1\overline{u}\right] \\ &\quad + \frac{D\left(p_{LS}\right)}{z_{LS}^2}\left[-\left(p_{LS}-c_p-h_1\right)z_{LS}g\left(z_{LS}\right)\right]\end{aligned}$$

and since $-\left(p_{LS}-c_p-h_1\right)\int_0^{z_{LS}^*} ug\left(u\right)du + c - h_1\overline{u} = 0$, we get

$$\frac{\partial^2 E\left[P_{E\_LS}\left(p_{LS}, z_{LS}\right)\right]}{\partial z_{LS}^2}\Big|_{\frac{\partial E\left[P_{E\_LS}\left(p_{LS}, z_{LS}\right)\right]}{\partial z_{LS}}=0} = \frac{D\left(p_{LS}\right)}{z_{LS}^2}\left[-\left(p_{LS}-c_p-h_1\right)z_{LS}g\left(z_{LS}\right)\right] < 0.$$

b) We know from the first-order derivative that $\left(p_{LS}-c_p-h_1\right)\int_0^{z_{LS}^*} ug\left(u\right)du = c - h_1\overline{u}$, and therefore as p_{LS} increases z_{LS}^* cannot increase in order to maintain the equality. Therefore, $z_{LS}^*\left(p_{LS}\right)$ is non-increasing in p_{LS}.

c) The first-order derivative of (30) with respect to p_{LS} for a given z_{LS} is:

$$\begin{aligned}\frac{\partial E\left[P_{E\_LS}\left(p_{LS}, z_{LS}\right)\right]}{\partial p_{LS}} &= \frac{D\left(p_{LS}\right)}{z_{LS}}\left(z_{LS} - \int_0^{z_{LS}}\left(z_{LS}-u\right)g\left(u\right)du\right) \\ &+ \left[\left(\frac{\partial D\left(p_{LS}\right)}{\partial p_{LS}}\right)\frac{1}{z_{LS}}\left[\left(p_{LS}-c_p-h_1\right)\left(z_{LS} - \int_0^{z_{LS}}\left(z_{LS}-u\right)g\left(u\right)du\right) - \left(c - h_1\overline{u}\right)\right]\right].\end{aligned}$$

It is already shown that $z_{LS}^* = 1$ in the price range $p_{LS} < c_p + \frac{c}{\overline{u}}$. The first-order derivative of the objective function with respect to p_{LS} is:

$$\frac{\partial E\left[P_{E\_LS}\left(p_{LS}, z_{LS}=1\right)\right]}{\partial p_{LS}} = \left(\frac{\partial D\left(p_{LS}\right)}{\partial p_{LS}}\right)\left(\overline{u}\right)\left[\frac{D\left(p_{LS}\right)}{\left(\frac{\partial D\left(p_{LS}\right)}{\partial p_{LS}}\right)} + \left(p_{LS} - c_p - \frac{c}{\overline{u}}\right)\right] > 0.$$

Thus, the objective function improves as the sale price increases to $p_{LS} \leq c_p + \frac{c}{\overline{u}}$. Therefore, we consider the behavior of the objective function for price values $p_{LS} > c_p + \frac{c}{\overline{u}}$. The second-order derivative of (30) with respect to p_{LS} for a given z_{LS} is:

$$\begin{aligned}\frac{\partial^2 E\left[P_{E\_LS}\left(p_{LS}, z_{LS}\right)\right]}{\partial p_{LS}^2} &= 2\left(\frac{\partial D\left(p_{LS}\right)}{\partial p_{LS}}\right)\frac{1}{z_{LS}}\left(z_{LS} - \int_0^{z_{LS}}\left(z_{LS}-u\right)g\left(u\right)du\right) \\ &\quad + \left(\frac{\partial^2 D\left(p_{LS}\right)}{\partial p_{LS}^2}\right)\frac{1}{z_{LS}}\left[\left(p_{LS}-c_p-h_1\right)\left(z_{LS} - \int_0^{z_{LS}}\left(z_{LS}-u\right)g\left(u\right)du\right) - \left(c - h_1\overline{u}\right)\right].\end{aligned}$$

When demand is linear in price as in (1), $\frac{\partial D\left(p_{LS}\right)}{\partial p_{LS}} = -b$ and $\frac{\partial^2 D\left(p_{LS}\right)}{\partial p_{LS}^2} = 0$. Therefore,

$$\frac{\partial^2 E\left[P_{E\_LS}\left(p_{LS}, z_{LS}\right)\right]}{\partial p_{LS}^2} = 2\left(-b\right)\frac{1}{z_{LS}}\left(z_{LS} - \int_0^{z_{LS}}\left(z_{LS}-u\right)g\left(u\right)du\right) < 0.$$

Thus, the objective function (30) is concave in p_{LS} for a given z_{LS}, and the optimal $p_{LS}^*\left(z_{LS}\right)$ is unique and is equal to:

$$p_{LS}^*\left(z_{LS}\right) = \frac{a + b\left(c_p + h_1 + \frac{c - h_1\overline{u}}{\left[z_{LS} - \int_0^{z_{LS}} G(u)du\right]}\right)}{2b}$$

Rewriting the first-order derivative of (30) with respect to p_{LS} for a given z_{LS} helps prove the same result under iso-elastic demand.

$$\frac{\partial E\left[P_{E\_LS}\left(p_{LS}, z_{LS}\right)\right]}{\partial p_{LS}} = \frac{\left(\frac{\partial D(p_{LS})}{\partial p_{LS}}\right)\left(z_{LS}-\int_0^{z_{LS}}\left(z_{LS}-u\right) g\left(u\right) du\right)}{z_{LS}} R_p\left(p_{LS}, z_{LS}\right) \text{ where}$$

$$R_p\left(p_{LS}, z_{LS}\right) = \left[\begin{array}{c} \frac{D(p_{LS})}{\left(\frac{\partial D(p_{LS})}{\partial p_{LS}}\right)} + \left(p_{LS}-c_p-h_1\right) \\ -\frac{(c-h_1\overline{u})}{\left(z_{LS}-\int_0^{z_{LS}}(z_{LS}-u)g(u)du\right)} \end{array}\right]$$

Thus, $\frac{\partial E\left[P_{E\_LS}(p_{LS},z_{LS})\right]}{\partial p_{LS}}=0$ means $\left[\frac{D(p_{LS})}{\left(\frac{\partial D(p_{LS})}{\partial p_{LS}}\right)}+\left(p_{LS}-c_p-h_1\right)-\frac{(c-h_1\overline{u})}{\left(z_{LS}-\int_0^{z_{LS}}(z_{LS}-u)g(u)du\right)}\right]=0$. It should be observed that when demand is iso-elastic in price as in (2), we have $\frac{\partial D(p_{LS})}{\partial p_{LS}}=-\frac{b}{p_{LS}}D\left(p_{LS}\right)<0$, and $\frac{D(p_{LS})}{\left(\frac{\partial D(p_{LS})}{\partial p_{LS}}\right)}= -\frac{p_{LS}}{b}$; therefore $\frac{D(p_{LS})}{\left(\frac{\partial D(p_{LS})}{\partial p_{LS}}\right)}+p_{LS}=\left(\frac{b-1}{b}\right)p$. Then, we have $\frac{\partial R_p(p_{LS},z_{LS})}{\partial p_{LS}}=\left(\frac{b-1}{b}\right)>0$, and $R_p\left(p_{LS}, z_{LS}\right)$ is increasing in p_{LS} for a given z_{LS}, which implies that $\frac{\partial E\left[P_{E\_LS}(p_{LS},z_{LS})\right]}{\partial p_{LS}}$ is positive when $R_p\left(p_{LS}, z_{LS}\right)<0$ and $\frac{\partial E\left[P_{E\_LS}(p_{LS},z_{LS})\right]}{\partial p_{LS}}$ is negative when $R_p\left(p_{LS}, z_{LS}\right)>0$. This means that the sign of $\frac{\partial E\left[P_{E\_LS}(p_{LS},z_{LS})\right]}{\partial p_{LS}}$ switches from positive to negative as p_{LS} increases. As a result, $\left[\frac{D(p_{LS})}{\left(\frac{\partial D(p_{LS})}{\partial p_{LS}}\right)}+\left(p_{LS}-c_p-h_1\right)-\frac{(c-h_1\overline{u})}{\left(z_{LS}-\int_0^{z_{LS}}(z_{LS}-u)g(u)du\right)}\right]= 0$ provides the unique optimal solution to the objective function for a given z_{LS}:

$$p_{LS}\left(z_{LS}\right)=\left(\frac{b}{b-1}\right)\left(c_p+h_1+\frac{(c-h_1\overline{u})}{\left(z_{LS}-\int_0^{z_{LS}}\left(z_{LS}-u\right)g\left(u\right)du\right)}\right).$$

In order to establish that $z^*_{LS}\left(p_{LS}\right)$ is unique, we substitute the optimal sale price expression for a given z_{LS} obtained through equations (33) and (34) into $\frac{\partial E\left[P_{E\_LS}(p_{LS},z_{LS})\right]}{\partial z_{LS}}=0$:

$$\frac{\partial E\left[P_{E\_LS}\left(p^*_{LS}\left(z_{LS}\right), z_{LS}\right)\right]}{\partial z_{LS}} = \frac{D\left(p^*_{LS}\left(z_{LS}\right)\right)}{z^2_{LS}}\left[-\left(p^*_{LS}\left(z_{LS}\right)-c_p-h_1\right)\int_0^{z_{LS}} ug\left(u\right)du+c-h_1\overline{u}\right]=0$$

We next define $R\left(z_{LS}\right)=-\left(p^*_{LS}\left(z_{LS}\right)-c_p-h_1\right)\int_0^{z_{LS}} ug\left(u\right)du+c-h_1\overline{u}$, and show that $\frac{\partial R(z)}{\partial z_{LS}}<0$ for both linear and iso-elastic demand.

$$\begin{aligned} R\left(z_{LS}\right) &= -\left(p^*_{LS}\left(z_{LS}\right)-c_p-h_1\right)\int_0^{z_{LS}} ug\left(u\right)du+c-h_1\overline{u} \\ &= \left(p^*_{LS}\left(z_{LS}\right)-c_p-h_1\right)\left[z_{LS}\left(1-G\left(z_{LS}\right)\right)-\left(z_{LS}-\int_0^{z_{LS}}\left(z_{LS}-u\right)g\left(u\right)du\right)\right]+c-h_1\overline{u} \end{aligned}$$

Define $\delta\left(z_{LS}\right)=\frac{z_{LS}(1-G(z_{LS}))}{\left(z_{LS}-\int_0^{z_{LS}}(z_{LS}-u)g(u)du\right)}$ where $\delta\left(z_{LS}\right)<1$, and it is known from Proposition 2.b of Petruzzi (2008) that $\frac{\partial \delta(z_{LS})}{\partial z_{LS}}<0$ when $g\left(\cdot\right)$ is IGFR. Then,

$$R\left(z_{LS}\right)=\left(p^*_{LS}\left(z_{LS}\right)-c_p-h_1\right)\left(z_{LS}-\int_0^{z_{LS}}\left(z_{LS}-u\right)g\left(u\right)du\right)\left(\delta\left(z_{LS}\right)-1\right)+c-h_1\overline{u}$$

For linear demand, we have $p_{LS}^{*}(z_{LS}) = \frac{a+b\left(c_p+h_1+\frac{c-h_1\overline{u}}{\left[z_{LS}-\int_0^{z_{LS}} G(u)du\right]}\right)}{2b}$, and

$$\begin{aligned} R(z_{LS}) &= \left(\frac{a+b\left(c_p+h_1+\frac{c-h_1\overline{u}}{\left[z_{LS}-\int_0^{z_{LS}}(z_{LS}-u)g(u)du\right]}\right)}{2b} - c_p - h_1\right)\left(z_{LS} - \int_0^{z_{LS}} (z_{LS}-u)\,g(u)\,du\right)(\delta(z_{LS})-1) \\ &\quad + c - h_1\overline{u} \\ &= \left(\frac{a}{2b} - \frac{(c_p+h_1)}{2} + \frac{c-h_1\overline{u}}{2\left[z_{LS}-\int_0^{z_{LS}}(z_{LS}-u)\,g(u)\,du\right]}\right)\left(z_{LS} - \int_0^{z_{LS}} (z_{LS}-u)\,g(u)\,du\right)(\delta(z_{LS})-1) \\ &\quad + c - h_1\overline{u} \\ &= \left(\left(\frac{a}{2b} - \frac{(c_p+h_1)}{2}\right)\left(z_{LS} - \int_0^{z_{LS}} (z_{LS}-u)\,g(u)\,du\right) + \frac{c-h_1\overline{u}}{2}\right)(\delta(z_{LS})-1) + c - h_1\overline{u} \end{aligned}$$

Note that $\frac{\partial\left[(p_{LS}^{*}(z_{LS})-c_p-h_1)\left(z_{LS}-\int_0^{z_{LS}}(z_{LS}-u)g(u)du\right)\right]}{\partial z_{LS}} = \left(\frac{a}{2b} - \frac{(c_p+h_1)}{2}\right)(1-G(z_{LS})) > 0$ for linear demand, and

$$\begin{aligned} \frac{\partial R(z_{LS})}{\partial z_{LS}} &= \left(\left(\frac{a}{2b} - \frac{(c_p+h_1)}{2}\right)\left(z_{LS} - \int_0^{z_{LS}} (z_{LS}-u)\,g(u)\,du\right) + \frac{c-h_1\overline{u}}{2}\right)\left(\frac{\partial\delta(z_{LS})}{\partial z_{LS}}\right) \\ &\quad + (\delta(z_{LS})-1)\frac{\partial\left[(p_{LS}^{*}(z_{LS})-c_p-h_1)\left(z_{LS}-\int_0^{z_{LS}}(z_{LS}-u)\,g(u)\,du\right)\right]}{\partial z_{LS}}. \end{aligned}$$

Because $\left(\frac{\partial\delta(z_{LS})}{\partial z_{LS}}\right) < 0$ the first term is negative; and, because $\frac{\partial\left[(p_{LS}^{*}(z_{LS})-c_p-h_1)\left(z_{LS}-\int_0^{z_{LS}}(z_{LS}-u)g(u)du\right)\right]}{\partial z_{LS}} > 0$ and $\delta(z_{LS}) < 1$ the second term is also negative. Therefore, $\frac{\partial R(z_{LS})}{\partial z_{LS}} < 0$ for linear demand.

For iso-elastic demand, we have $p_{LS}^{*}(z_{LS}) = \frac{b}{(b-1)}\left(c_p + h_1 + \frac{c-h_1\overline{u}}{\left[z_{LS}-\int_0^{z_{LS}} G(u)du\right]}\right)$, and

$$\begin{aligned} R(z_{LS}) &= \left(\frac{b}{(b-1)}\left(c_p+h_1+\frac{c-h_1\overline{u}}{\left[z_{LS}-\int_0^{z_{LS}} G(u)\,du\right]}\right) - c_p - h_1\right)\left(z_{LS} - \int_0^{z_{LS}} (z_{LS}-u)\,g(u)\,du\right)(\delta(z_{LS})-1) \\ &\quad + c - h_1\overline{u} \\ &= \left(\frac{1}{(b-1)}(c_p+h_1)\left(z_{LS} - \int_0^{z_{LS}} (z_{LS}-u)\,g(u)\,du\right) + \frac{b}{(b-1)}(c-h_1\overline{u})\right)(\delta(z_{LS})-1) \\ &\quad + c - h_1\overline{u}. \end{aligned}$$

Note that $\frac{\partial\left[(p_{LS}^{*}(z_{LS})-c_p-h_1)\left(z_{LS}-\int_0^{z_{LS}}(z_{LS}-u)g(u)du\right)\right]}{\partial z_{LS}} = \left(\frac{1}{(b-1)}(c_p+h_1)\right)(1-G(z_{LS})) > 0$ for iso-elastic demand, and

$$\begin{aligned} \frac{\partial R(z_{LS})}{\partial z_{LS}} &= \left(\frac{1}{(b-1)}(c_p+h_1)\left(z_{LS} - \int_0^{z_{LS}} (z_{LS}-u)\,g(u)\,du\right) + \frac{b}{(b-1)}(c-h_1\overline{u})\right)\left(\frac{\partial\delta(z_{LS})}{\partial z_{LS}}\right) \\ &\quad + (\delta(z_{LS})-1)\frac{\partial\left[(p_{LS}^{*}(z_{LS})-c_p-h_1)\left(z_{LS}-\int_0^{z_{LS}}(z_{LS}-u)\,g(u)\,du\right)\right]}{\partial z_{LS}}. \end{aligned}$$

Because $\left(\frac{\partial\delta(z_{LS})}{\partial z_{LS}}\right) < 0$, the first term is negative; and, because $\frac{\partial\left[(p_{LS}^{*}(z_{LS})-c_p-h_1)\left(z_{LS}-\int_0^{z_{LS}}(z_{LS}-u)g(u)du\right)\right]}{\partial z_{LS}} > 0$ and $\delta(z_{LS}) < 1$, the second term is also negative. Therefore, $\frac{\partial R(z_{LS})}{\partial z_{LS}} < 0$ for iso-elastic demand.

d) The derivative of the optimal sale price expression with respect to z_{LS} is negative, i.e., $\frac{\partial p_{LS}^{*}(z_{LS})}{\partial z_{LS}} = -\frac{(c-h_1\overline{u})(1-G(z))}{\left[z_{LS}-\int_0^{z_{LS}} G(u)du\right]^2} \leq 0$. Therefore, $p_{LS}^{*}(z_{LS})$ is non-increasing in z_{LS}.

e) In the analysis of deterministic supply using the expected unit cost of growing, the firm has $p_g^0 = c_p + \frac{c}{\overline{u}} - \left(\frac{D(p)}{\frac{\partial D(p)}{\partial p}}\right)$. Observe that

$$\left(z_{LS} - \int_0^{z_{LS}} (z_{LS}-u)\,g(u)\,du\right) = \overline{u} - \int_{z_{LS}}^{1} (u-z_{LS})\,g(u)\,du \leq \overline{u}$$

and therefore, the optimal sale price in (32) is:

$$\begin{aligned} p_{LS}^{*}(z_{LS}) &= c_p + h_1 + \frac{c - h_1\overline{u}}{\left[z_{LS} - \int_0^{z_{LS}} (z_{LS} - u)\, g(u)\, du\right]} - \left(\frac{D(p)}{\left(\frac{\partial D(p)}{\partial p}\right)}\right) \\ &\geq c_p + h_1 + \frac{c - h_1\overline{u}}{\overline{u}} - \left(\frac{D(p)}{\left(\frac{\partial D(p)}{\partial p}\right)}\right) = c_p + \frac{c}{\overline{u}} - \left(\frac{D(p)}{\left(\frac{\partial D(p)}{\partial p}\right)}\right) = p_g^0. \end{aligned}$$

Proof of Proposition 4:

In the region $p \leq c_p + c_2(u=1)$, the firm would not engage in the purchasing option and $q_2^* = 0$ in all realizations of u. Thus, the problem converges to the lost sales variant in this price range. This policy can be perceived as the optimal solution to the problem (EP) with the price constraint $p \leq c_p + c_2(u=1)$.

a) Comparing (26) and (31), which can also be expressed as $\int_0^{z_{LS}^*} (p_{LS} - c_p - h_1)\, ug(u)\, du = c - h_1\overline{u}$, provides the result. Both stocking level expressions have the same right-hand side value of $(c - h_1\overline{u})$. Because $c_2(u) - h_1 > p - c_p - h_1$ for each realization of u, the stocking level that satisfies the right-hand side of (26) has to be less than that of (31); thus, $z_{LS}^*(p_{\mathrm{I}}) = z_I^*(p_{\mathrm{I}}) > z_{CB}^*$ for any price $p_{\mathrm{I}} \leq c_p + c_2(u=1)$.

b) Because increasing q_2 decreases the objective function by $c_p + c_2(u) - p_{\mathrm{I}} > 0$ for each realization of u, we know that $E[P_E(p_{\mathrm{I}}, z_{\mathrm{I}})] > E\left[P_{E\_CB}(p_{\mathrm{I}}, z_{\mathrm{I}})\right]$ for all pairs of $(p_{\mathrm{I}}, z_{\mathrm{I}})$, and thus, $E[P_E(p_{\mathrm{I}}, z_{CB}^*)] > E\left[P_{E\_CB}(p_{\mathrm{I}}, z_{CB}^*)\right]$ for any price p_{I}.

c) If the optimal price of lost sales is in the same range, then $z_{\mathrm{I}}^* = z_{LS}^*$ and $E[P_E(p_{\mathrm{I}}^*, z_{\mathrm{I}}^*)] = E\left[P_{E\_LS}(p_{LS}^*, z_{LS}^*)\right]$.

d) If the optimal sale price is greater than $c_p + c_2(u=1)$, then we already know that $z_{LS}(p)$ is non-increasing in p, and thus $z_{\mathrm{I}}^*(p_{\mathrm{I}}) = z_{LS}^*(p_{\mathrm{I}}) \geq z_{LS}^*(p_{LS}^*)$. Furthermore, we have $E\left[P_{E\_LS}(p_{LS}^*, z_{LS}^*(p_{LS}^*))\right] > E\left[P_{E\_LS}(p_{\mathrm{I}}^*, z_{\mathrm{I}}^*(p_{\mathrm{I}}^*))\right] = E[P_E(p_{\mathrm{I}}^*, z_{\mathrm{I}}^*(p_{\mathrm{I}}^*))]$, eliminating the possibility of the optimal choice for the problem (EP) being located in region I.

e) The proof follows from part d) and Proposition 3.e). Because $p_{LS}^* > p_g^0 > c_p + \frac{c}{\overline{u}}$ according to Proposition 3, when $\frac{c}{\overline{u}} > c_2(u=1)$ the firm has $p_{LS}^* > c_p + c_2(u=1)$, and the optimal solution cannot be located in region I. Note that $z_{LS}^*\left(p = c_p + \frac{c}{\overline{u}}\right) = 1$ and because $z_{LS}^*(p)$ is non-increasing in p, $z_{LS}^*(p) = 1$ for all values of $p \leq c_p + \frac{c}{\overline{u}}$. Because (3) increases with a higher value of p when $z = 1$, resulting in the need to increase the sale price beyond $c_p + c_2(u=1)$. Thus, the optimal solution cannot be in region I.

Proof of Proposition 5:

We consider region II when $c_p + c_2(u=1) < p_{\mathrm{II}} < c_p + c_2(u=0)$. Using the transformation $z_{\mathrm{II}} \equiv \frac{D(p_{\mathrm{II}})}{Q}$, the objective function in (3) can be written as follows:

$$\begin{aligned} E[P_E(p_{\mathrm{II}}, z_{\mathrm{II}})] &= \frac{D(p_{\mathrm{II}})}{z_{\mathrm{II}}} \left\{ \begin{array}{c} \left(p_{\mathrm{II}} - c_p - \frac{c}{\overline{u}}\right) z_{\mathrm{II}} - \left(p_{\mathrm{II}} - c_p - \frac{c}{\overline{u}}\right) \int_0^{c_2^{-1}(p_{\mathrm{II}} - c_p)} (z_{\mathrm{II}} - u)\, g(u)\, du \\ - \int_{c_2^{-1}(p_{\mathrm{II}} - c_p)}^{z_{\mathrm{II}}} \left(c_2(u) - \frac{c}{\overline{u}}\right)(z_{\mathrm{II}} - u)\, g(u)\, du - \left(\frac{c}{\overline{u}} - h_1\right) \int_{z_{\mathrm{II}}}^1 (u - z_{\mathrm{II}})\, g(u)\, du \end{array} \right\} \\ &= \frac{D(p_{\mathrm{II}})}{z_{\mathrm{II}}} \left\{ \begin{array}{c} (p_{\mathrm{II}} - c_p - h_1)\, z_{\mathrm{II}} - (p_{\mathrm{II}} - c_p - h_1) \int_0^{c_2^{-1}(p_{\mathrm{II}} - c_p)} (z_{\mathrm{II}} - u)\, g(u)\, du \\ - \int_{c_2^{-1}(p_{\mathrm{II}} - c_p)}^{z_{\mathrm{II}}} (c_2(u) - h_1)(z_{\mathrm{II}} - u)\, g(u)\, du - (c - h_1\overline{u}) \end{array} \right\}. \end{aligned}$$

a) The first-order derivative of the objective function w.r.t. z_{II} for a given p_{II} is as follows:

$$\frac{\partial E\left[P_E\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)\right]}{\partial z_{\mathrm{II}}} = \frac{D\left(p_{\mathrm{II}}\right)}{z_{\mathrm{II}}^2} \times R\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right) \text{ where}$$

$$R\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right) = \left\{ \begin{array}{c} -\left(p_{\mathrm{II}}-c_p-h_1\right) \int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)} u g(u)\, du \\ -\int_{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}^{z_{\mathrm{II}}}\left(c_2(u)-h_1\right) u g(u)\, du+\left(c-h_1 \overline{u}\right) \end{array} \right\}.$$

Note that at the boundary point $z_{\mathrm{II}}=0$, $R\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)=\left(c-h_1 \overline{u}\right)>0$, implying that increasing z_{II} would increase the objective function value. Next consider the other boundary point $z_{\mathrm{II}}=1$. The value of $R\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)$ varies with the value of $c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)$. The point $c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)=1$ implies that $p_{\mathrm{II}}=c_p+c_2(u=1)$, and at this point, $R\left(p_{\mathrm{II}}, z_{\mathrm{II}}=1\right)=-\left(c_2(u=1)-h_1\right) \overline{u}+\left(c-h_1 \overline{u}\right)=-c_2(u=1) \overline{u}+c$. Note that if $c_2(u=1)<\frac{c}{\overline{u}}$, then $R\left(p_{\mathrm{II}}, z_{\mathrm{II}}=1\right)>0$, which implies that at $p_{\mathrm{II}}=c_p+c_2^{-1}(u=1)$, $z_{\mathrm{II}}^*=1$. However, if $c_2(u=1) \geq \frac{c}{\overline{u}}$, then $R\left(p_{\mathrm{II}}, z_{\mathrm{II}}=1\right)<0$, which implies that $z_{\mathrm{II}}^*<1$. In the case of $c_2(u=1)<\frac{c}{\overline{u}}$, there exists a minimum value of p_{II} denoted by $p_{\mathrm{II}}^{\min}$ where $R\left(p_{\mathrm{II}}=p_{\mathrm{II}}^{\min}, z_{\mathrm{II}}=1\right)=0$, and $p_{\mathrm{II}}^{\min}$ satisfies:

$$\left(p_{\mathrm{II}}^{\min}-c_p\right) \int_0^{c_2^{-1}\left(p_{\mathrm{II}}^{\min}-c_p\right)} u g(u)\, du+\int_{c_2^{-1}\left(p_{\mathrm{II}}^{\min}-c_p\right)}^1 c_2(u) u g(u)\, du=c$$

For the price values of $c_p+c_2^{-1}(u=1)<p_{\mathrm{II}} \leq p_{\mathrm{II}}^{\min}$, $R\left(p_{\mathrm{II}}, z_{\mathrm{II}}=1\right)>0$, which implies that $z_{\mathrm{II}}^*=1$. For the price values $p_{\mathrm{II}}^{\min}<p_{\mathrm{II}} \leq c_p+c_2^{-1}(u=0)$, $R\left(p_{\mathrm{II}}, z_{\mathrm{II}}=1\right)<0$, which implies that $z_{\mathrm{II}}^*<1$. The point $c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)=0$ implies that $p_{\mathrm{II}}=c_p+c_2^{-1}(u=0)$, and at this point, $R\left(p_{\mathrm{II}}, z_{\mathrm{II}}=1\right)=-\int_0^1 c_2(u) u g(u)\, du+c<0$ by the assumption that the expected purchasing cost is more expensive than the expected growing cost. Next, consider the case of p_{II} such that $c_p+c_2(u=1)<p_{\mathrm{II}}<c_p+c_2(u=0)$, which in turn means $0<c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)<1$. In this case,

$$\frac{\partial E\left[P_E\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)\right]}{\partial z_{\mathrm{II}}} \Big|_{\frac{\partial E\left[P_E\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)\right]}{\partial z}=0}=-\frac{D\left(p_{\mathrm{II}}\right)}{z_{\mathrm{II}}^2}\left(c_2\left(z_{\mathrm{II}}\right)-h_1\right) z_{\mathrm{II}} g\left(z_{\mathrm{II}}\right)<0.$$

This implies that $\frac{\partial E\left[P_E\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)\right]}{\partial z_{\mathrm{II}}}$ is positive for smaller values of z_{II}, and it continues to decrease once it starts decreasing. Therefore, $\frac{\partial E\left[P_E\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)\right]}{\partial z_{\mathrm{II}}}$ decreases from positive to negative as z_{II} goes from 0 to 1. As a result, $\frac{\partial E\left[P_E\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)\right]}{\partial z_{\mathrm{II}}}=0$ is a unique maximizer. The optimal value of z_{II}^* is the value that satisfies:

$$\int_{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}^{z_{\mathrm{II}}^*}\left(c_2(u)-h_1\right) u g(u)\, du=\left(c-h_1 \overline{u}\right)-\left(p_{\mathrm{II}}-c_p-h_1\right) \int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)} u g(u)\, du.$$

b) Consider the solution that satisfies $\frac{\partial E\left[P_E\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)\right]}{\partial z_{\mathrm{II}}}=0$, which requires that $\left\{ \begin{array}{c} -\left(p_{\mathrm{II}}-c_p-h_1\right) \int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)} u g(u)\, du \\ -\int_{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}^{z_{\mathrm{II}}}\left(c_2(u)-h_1\right) u g(u)\, du+\left(c-h_1 \overline{u}\right) \end{array} \right\}=0$. Taking the derivative of $\frac{\partial\left\{ \begin{array}{c} -\left(p_{\mathrm{II}}-c_p-h_1\right) \int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)} u g(u)\, du \\ -\int_{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}^{z_{\mathrm{II}}}\left(c_2(u)-h_1\right) u g(u)\, du+\left(c-h_1 \overline{u}\right) \end{array} \right\}}{\partial p_{\mathrm{II}}}=-\int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)} u g(u)\, du<0$. As a result, in order to maintain $\frac{\partial E\left[P_E\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)\right]}{\partial z_{\mathrm{II}}}=0$, $z_{\mathrm{II}}^*\left(p_{\mathrm{II}}\right)$ must be non-increasing in p_{II}.

c) The first-order derivative of the objective function w.r.t. p_{II} for a given z_{II} is as follows:

$$\begin{aligned}
\frac{\partial E\left[P_E\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)\right]}{\partial p_{\mathrm{II}}} &= \frac{D\left(p_{\mathrm{II}}\right)}{z_{\mathrm{II}}}\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}\left(z_{\mathrm{II}}-u\right) g(u)\, du\right) \\
&\quad +\frac{\left(\frac{\partial D\left(p_{\mathrm{II}}\right)}{\partial p_{\mathrm{II}}}\right)}{z_{\mathrm{II}}}\left\{\begin{array}{c}
\left(p_{\mathrm{II}}-c_p-h_1\right) z_{\mathrm{II}}-\left(p_{\mathrm{II}}-c_p-h_1\right) \int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}\left(z_{\mathrm{II}}-u\right) g(u)\, du \\
-\int_{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}^{z_{\mathrm{II}}}\left(c_2(u)-h_1\right)\left(z_{\mathrm{II}}-u\right) g(u)\, du-\left(c-h_1 \overline{u}\right)
\end{array}\right\} \\
&= \frac{\left(\frac{\partial D\left(p_{\mathrm{II}}\right)}{\partial p_{\mathrm{II}}}\right)}{z_{\mathrm{II}}}\left\{\begin{array}{c}
\left(\frac{D\left(p_{\mathrm{II}}\right)}{\left(\frac{\partial D\left(p_{\mathrm{II}}\right)}{\partial p_{\mathrm{II}}}\right)}+p_{\mathrm{II}}-c_p-h_1\right)\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}(z-u) g(u)\, du\right) \\
-\int_{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}^{z_{\mathrm{II}}}\left(c_2(u)-h_1\right)\left(z_{\mathrm{II}}-u\right) g(u)\, du-\left(c-h_1 \overline{u}\right)
\end{array}\right\} \\
&= \frac{\left(\frac{\partial D\left(p_{\mathrm{II}}\right)}{\partial p_{\mathrm{II}}}\right)}{z_{\mathrm{II}}} R_p\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)
\end{aligned}$$

As we show later $R_p\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)$ is increasing in p_{II} for a given z_{II}. When $R_p\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)$ is increasing in p_{II} for a given z_{II}, equating $\frac{\partial E\left[P_E\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)\right]}{\partial p_{\mathrm{II}}}$ to zero implies that $R_p\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)=0$ and provides the optimal choice for p_{II}:

$$p_{\mathrm{II}}^*\left(z_{\mathrm{II}}\right)=c_p+h_1+\frac{\int_{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}^{z_{\mathrm{II}}}\left(c_2(u)-h_1\right)\left(z_{\mathrm{II}}-u\right) g(u)\, du}{\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}\left(z_{\mathrm{II}}-u\right) g(u)\, du\right)}+\frac{\left(c-h_1 \overline{u}\right)}{\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}\left(z_{\mathrm{II}}-u\right) g(u)\, du\right)}-\frac{D\left(p_{\mathrm{II}}\right)}{\left(\frac{\partial D\left(p_{\mathrm{II}}\right)}{\partial p_{\mathrm{II}}}\right)}$$

$$p_{\mathrm{II}}^*\left(z_{\mathrm{II}}\right)\left(1-\frac{1}{\in\left(p_{\mathrm{II}}^*\left(z_{\mathrm{II}}\right)\right)}\right)=c_p+h_1+\frac{\int_{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}^{z_{\mathrm{II}}}\left(c_2(u)-h_1\right)\left(z_{\mathrm{II}}-u\right) g(u)\, du}{\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}\left(z_{\mathrm{II}}-u\right) g(u)\, du\right)}+\frac{\left(c-h_1 \overline{u}\right)}{\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}\left(z_{\mathrm{II}}-u\right) g(u)\, du\right)}.$$

Before proceeding with the analysis of $R_p\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)$, observe that the ratio $\frac{\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}\left(z_{\mathrm{II}}-u\right) g(u) du\right)}{\left(z_{\mathrm{II}}-c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right) g\left(c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right)}$ is positive, and is decreasing in $z_{\mathrm{II}}>c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)$ for a given p_{II}, and obtains its lowest value at $z_{\mathrm{II}}=1$. To see this,

$$\begin{aligned}
\frac{\partial\left(\frac{\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}\left(z_{\mathrm{II}}-u\right) g(u) du\right)}{\left(z_{\mathrm{II}}-c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right) g\left(c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right)}\right)}{\partial z_{\mathrm{II}}} &= \frac{\left[\begin{array}{c}
\left(1-G\left(c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right)\right)\left(z_{\mathrm{II}}-c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right) g\left(c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right) \\
-g\left(c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right)\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}\left(z_{\mathrm{II}}-u\right) g(u)\, du\right)
\end{array}\right]}{\left(\left(z_{\mathrm{II}}-c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right) g\left(c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right)\right)^2} \\
&= \frac{-\left[\begin{array}{c}
c_2^{-1}\left(p_{\mathrm{II}}-c_p\right) \\
-\int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}\left(c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)-u\right) g(u)\, du
\end{array}\right] g\left(c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right)}{\left(\left(z_{\mathrm{II}}-c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right) g\left(c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right)\right)^2} \leq 0,
\end{aligned}$$

and $\frac{\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}\left(z_{\mathrm{II}}-u\right) g(u) du\right)}{\left(z_{\mathrm{II}}-c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right) g\left(c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right)}$ is at its minimum when $z_{\mathrm{II}}=1$, which equals to $\frac{\left(1-\int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}(1-u) g(u) du\right)}{\left(1-c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right) g\left(c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right)}$. Observe that $\frac{\left(1-\int_0^{c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)}(1-u) g(u) du\right)}{\left(1-c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right) g\left(c_2^{-1}\left(p_{\mathrm{II}}-c_p\right)\right)}>\overline{u}$.

Let us next take the first-order derivative of $R_p\left(p_{\mathrm{II}}, z_{\mathrm{II}}\right)$ in order to show that it is increasing in p_{II} for a given

z_{II}.

$$\begin{aligned}\frac{\partial R_p(p_{\mathrm{II}},z_{\mathrm{II}})}{\partial p_{\mathrm{II}}} &= \left(\frac{\left(\frac{\partial D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}}\right)^2-\left(\frac{\partial^2 D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}^2}\right)D(p_{\mathrm{II}})}{\left(\frac{\partial D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}}\right)^2}+1\right)\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}(p_{\mathrm{II}}-c_p)}(z_{\mathrm{II}}-u)\,g(u)\,du\right)\\
&\quad-\left(\frac{D(p_{\mathrm{II}})}{\left(\frac{\partial D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}}\right)}\right)\left(z_{\mathrm{II}}-c_2^{-1}(p_{\mathrm{II}}-c_p)\right)g\left(c_2^{-1}(p_{\mathrm{II}}-c_p)\right)\left(\frac{\partial c_2^{-1}(p_{\mathrm{II}}-c_p)}{\partial p_{\mathrm{II}}}\right)\\
&= \left(2-\frac{\left(\frac{\partial^2 D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}^2}\right)D(p_{\mathrm{II}})}{\left(\frac{\partial D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}}\right)^2}\right)\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}(p_{\mathrm{II}}-c_p)}(z_{\mathrm{II}}-u)\,g(u)\,du\right)\\
&\quad-\left(\frac{D(p_{\mathrm{II}})}{\left(\frac{\partial D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}}\right)}\right)\left(\frac{\partial c_2^{-1}(p_{\mathrm{II}}-c_p)}{\partial p_{\mathrm{II}}}\right)\left(z_{\mathrm{II}}-c_2^{-1}(p_{\mathrm{II}}-c_p)\right)g\left(c_2^{-1}(p_{\mathrm{II}}-c_p)\right)\end{aligned}$$

Note that the term $-\left(\frac{D(p_{\mathrm{II}})}{\left(\frac{\partial D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}}\right)}\right)\left(\frac{\partial c_2^{-1}(p_{\mathrm{II}}-c_p)}{\partial p_{\mathrm{II}}}\right)\left(z_{\mathrm{II}}-c_2^{-1}(p_{\mathrm{II}}-c_p)\right)g\left(c_2^{-1}(p_{\mathrm{II}}-c_p)\right)<0$, and $\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}(p_{\mathrm{II}}-c_p)}(z_{\mathrm{II}}-u)\,g(u)\,du\right)>0$.

$$\begin{aligned}\frac{\partial R_p(p_{\mathrm{II}},z_{\mathrm{II}})}{\partial p_{\mathrm{II}}} &= -\left(\frac{D(p_{\mathrm{II}})}{\left(\frac{\partial D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}}\right)}\right)\left(\begin{array}{c}z_{\mathrm{II}}-\\ \int_0^{c_2^{-1}(p_{\mathrm{II}}-c_p)}(z_{\mathrm{II}}-u)\,g(u)\,du\end{array}\right)\left\{\begin{array}{c}\left(-2\frac{\left(\frac{\partial D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}}\right)}{D(p_{\mathrm{II}})}+\frac{\left(\frac{\partial^2 D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}^2}\right)}{\left(\frac{\partial D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}}\right)}\right)+\\ \frac{\left(\frac{\partial c_2^{-1}(p_{\mathrm{II}}-c_p)}{\partial p_{\mathrm{II}}}\right)\left(z_{\mathrm{II}}-c_2^{-1}(p_{\mathrm{II}}-c_p)\right)g\left(c_2^{-1}(p_{\mathrm{II}}-c_p)\right)}{\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}(p_{\mathrm{II}}-c_p)}(z_{\mathrm{II}}-u)g(u)du\right)}\end{array}\right\}\\
&= \in(p)\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}(p_{\mathrm{II}}-c_p)}(z_{\mathrm{II}}-u)\,g(u)\,du\right)\left\{\begin{array}{c}\left(2\in(p)+\frac{p\left(\frac{\partial^2 D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}^2}\right)}{\left(\frac{\partial D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}}\right)}\right)+\\ \left(\frac{p\left(\frac{\partial c_2^{-1}(p_{\mathrm{II}}-c_p)}{\partial p_{\mathrm{II}}}\right)\left(z_{\mathrm{II}}-c_2^{-1}(p_{\mathrm{II}}-c_p)\right)g\left(c_2^{-1}(p_{\mathrm{II}}-c_p)\right)}{\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}(p_{\mathrm{II}}-c_p)}(z_{\mathrm{II}}-u)g(u)du\right)}\right)\end{array}\right\}\\
&= \in(p)\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}(p_{\mathrm{II}}-c_p)}(z_{\mathrm{II}}-u)\,g(u)\,du\right)\left\{\begin{array}{c}\left(2\in(p)+\frac{p\left(\frac{\partial^2 D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}^2}\right)}{\left(\frac{\partial D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}}\right)}\right)+\\ \frac{p\left(\frac{\partial c_2^{-1}(p_{\mathrm{II}}-c_p)}{\partial p_{\mathrm{II}}}\right)\left(z_{\mathrm{II}}-c_2^{-1}(p_{\mathrm{II}}-c_p)\right)g\left(c_2^{-1}(p_{\mathrm{II}}-c_p)\right)}{\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}(p_{\mathrm{II}}-c_p)}(z_{\mathrm{II}}-u)g(u)du\right)}\end{array}\right\}\end{aligned}$$

Therefore, when $\left(2\in(p)+\frac{p\left(\frac{\partial^2 D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}^2}\right)}{\left(\frac{\partial D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}}\right)}\right)+\frac{p\left(\frac{\partial c_2^{-1}(p_{\mathrm{II}}-c_p)}{\partial p_{\mathrm{II}}}\right)\left(z_{\mathrm{II}}-c_2^{-1}(p_{\mathrm{II}}-c_p)\right)g\left(c_2^{-1}(p_{\mathrm{II}}-c_p)\right)}{\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}(p_{\mathrm{II}}-c_p)}(z_{\mathrm{II}}-u)g(u)du\right)}>0$, then $\frac{\partial R_p(p_{\mathrm{II}},z_{\mathrm{II}})}{\partial p_{\mathrm{II}}}>0$.

$$\left(2\in(p)+\frac{p\left(\frac{\partial^2 D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}^2}\right)}{\left(\frac{\partial D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}}\right)}\right)+\frac{p\left(\frac{\partial c_2^{-1}(p_{\mathrm{II}}-c_p)}{\partial p_{\mathrm{II}}}\right)\left(z_{\mathrm{II}}-c_2^{-1}(p_{\mathrm{II}}-c_p)\right)g\left(c_2^{-1}(p_{\mathrm{II}}-c_p)\right)}{\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}(p_{\mathrm{II}}-c_p)}(z_{\mathrm{II}}-u)\,g(u)\,du\right)}>0$$

$$\frac{-p_{\mathrm{II}}\left(\frac{\partial c_2^{-1}(p_{\mathrm{II}}-c_p)}{\partial p_{\mathrm{II}}}\right)\left(z_{\mathrm{II}}-c_2^{-1}(p_{\mathrm{II}}-c_p)\right)g\left(c_2^{-1}(p_{\mathrm{II}}-c_p)\right)}{\left(z_{\mathrm{II}}-\int_0^{c_2^{-1}(p_{\mathrm{II}}-c_p)}(z_{\mathrm{II}}-u)\,g(u)\,du\right)}<2\in(p_{\mathrm{II}})+\frac{p_{\mathrm{II}}\left(\frac{\partial^2 D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}^2}\right)}{\left(\frac{\partial D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}}\right)}$$

The above inequality is the condition presented in (36). When demand is linear in price as in (1), $\frac{\partial D(p)}{\partial p}=-b$ and $\frac{\partial^2 D(p)}{\partial p^2}=0$, and thus the term $2\in(p_{\mathrm{II}})+\frac{p\left(\frac{\partial^2 D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}^2}\right)}{\left(\frac{\partial D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}}\right)}=\frac{2bp_{\mathrm{II}}}{a-bp_{\mathrm{II}}}>0$. When demand is iso-elastic as in (2),

$\frac{\partial D(p)}{\partial p} = -\frac{b}{p}D(p) < 0$, and $\frac{\partial^2 D(p)}{\partial p^2} = \frac{b(b+1)}{p^2}D(p) > 0$, and $\frac{D(p)}{\left(\frac{\partial D(p)}{\partial p}\right)} = -\frac{p}{b}$, and the term $2 \in (p_{\text{II}}) + \frac{p\left(\frac{\partial^2 D(p_{\text{II}})}{\partial p_{\text{II}}^2}\right)}{\left(\frac{\partial D(p_{\text{II}})}{\partial p_{\text{II}}}\right)} = b - 1 > 0$ because $b > 1$ by definition. Therefore, $\frac{\partial R_p(p_{\text{II}}, z_{\text{II}})}{\partial p_{\text{II}}}$ is positive when (36) is satisfied. As a result, for a given z_{II}, when the yield-dependent purchasing cost function satisfies (36), the optimal sale price can be obtained by equating $\frac{\partial E[P_E(p_{\text{II}}, z_{\text{II}})]}{\partial p_{\text{II}}} = 0$. When demand is linear as in (1), the optimal sale price satisfies the following:

$$p_{\text{II}}^*(z_{\text{II}}) = \frac{a + b\left(c_p + h_1 + \frac{-\int_{c_2^{-1}(p_{\text{II}}-c_p)}^{z_{\text{II}}}(c_2(u)-h_1)(z_{\text{II}}-u)g(u)du}{\left(z_{\text{II}} - \int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(z_{\text{II}}-u)g(u)du\right)} + \frac{(c - h_1\overline{u})}{\left(z_{\text{II}} - \int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(z_{\text{II}}-u)g(u)du\right)}\right)}{2b}.$$

When demand is iso-elastic as in (2), the optimal sale price satisfies the following:

$$p_{\text{II}}^*(z_{\text{II}}) = \frac{b}{(b-1)}\left(c_p + h_1 + \frac{-\int_{c_2^{-1}(p_{\text{II}}-c_p)}^{z_{\text{II}}}(c_2(u)-h_1)(z_{\text{II}}-u)g(u)\,du}{\left(z_{\text{II}} - \int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(z_{\text{II}}-u)g(u)\,du\right)} + \frac{(c - h_1\overline{u})}{\left(z_{\text{II}} - \int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(z_{\text{II}}-u)g(u)\,du\right)}\right).$$

d) Next, we show the uniqueness of the optimal solution. Consider the first-order derivative of the objective function w.r.t. z_{II}:

$$\frac{\partial E[P_E(p_{\text{II}}, z_{\text{II}})]}{\partial z_{\text{II}}} = \frac{D(p_{\text{II}})}{z_{\text{II}}^2} \times R(p_{\text{II}}, z_{\text{II}}) \text{ where}$$

$$R(p_{\text{II}}, z_{\text{II}}) = \left\{\begin{array}{c} -\left[\begin{array}{c}(p_{\text{II}} - c_p - h_1)z_{\text{II}} - \int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(p_{\text{II}} - c_p - h_1)(z_{\text{II}} - u)g(u)\,du \\ -\int_{c_2^{-1}(p_{\text{II}}-c_p)}^{z_{\text{II}}}(c_2(u) - h_1)(z_{\text{II}} - u)g(u)\,du\end{array}\right] \times \\ \left[1 - \frac{\left[\begin{array}{c}(p_{\text{II}} - c_p - h_1)z_{\text{II}} - \int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(p_{\text{II}} - c_p - h_1)z_{\text{II}}g(u)\,du \\ -\int_{c_2^{-1}(p_{\text{II}}-c_p)}^{z_{\text{II}}}(c_2(u) - h_1)z_{\text{II}}g(u)\,du\end{array}\right]}{\left[\begin{array}{c}(p_{\text{II}} - c_p - h_1)z_{\text{II}} - \int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(p_{\text{II}} - c_p - h_1)(z_{\text{II}} - u)g(u)\,du \\ -\int_{c_2^{-1}(p_{\text{II}}-c_p)}^{z_{\text{II}}}(c_2(u) - h_1)(z_{\text{II}} - u)g(u)\,du\end{array}\right]}\right] \\ +(c - h_1\overline{u})\end{array}\right\}.$$

It should be observed that

$$Ratio(p_{\text{II}}, z_{\text{II}}) = \frac{Num(p_{\text{II}}, z_{\text{II}})}{Den(p_{\text{II}}, z_{\text{II}})} = \frac{\left[\begin{array}{c}(p_{\text{II}} - c_p - h_1)z_{\text{II}} - \int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(p_{\text{II}} - c_p - h_1)z_{\text{II}}g(u)\,du \\ -\int_{c_2^{-1}(p_{\text{II}}-c_p)}^{z_{\text{II}}}(c_2(u) - h_1)z_{\text{II}}g(u)\,du\end{array}\right]}{\left[\begin{array}{c}(p_{\text{II}} - c_p - h_1)z_{\text{II}} - \int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(p_{\text{II}} - c_p - h_1)(z_{\text{II}} - u)g(u)\,du \\ -\int_{c_2^{-1}(p_{\text{II}}-c_p)}^{z_{\text{II}}}(c_2(u) - h_1)(z_{\text{II}} - u)g(u)\,du\end{array}\right]} \leq 1.$$

Next we prove that $Ratio(p_{\text{II}}, z_{\text{II}})$ is decreasing in z_{II} for a given p_{II}.

$$\begin{aligned} Num(p_{\text{II}}, z_{\text{II}}) &= \left[\begin{array}{c}(p_{\text{II}} - c_p - h_1)z_{\text{II}} - \int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(p_{\text{II}} - c_p - h_1)z_{\text{II}}g(u)\,du \\ -\int_{c_2^{-1}(p_{\text{II}}-c_p)}^{z_{\text{II}}}(c_2(u) - h_1)z_{\text{II}}g(u)\,du\end{array}\right] \\ &= z_{\text{II}}\left(\frac{\partial Den(p_{\text{II}}, z_{\text{II}})}{\partial z_{\text{II}}}\right) > 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial Num(p_{\text{II}}, z_{\text{II}})}{\partial z_{\text{II}}} &= \left[\begin{array}{c}(p_{\text{II}} - c_p - h_1) - \int_0^{c_2^{-1}(p_{\text{II}}-c_p)}(p_{\text{II}} - c_p - h_1)g(u)\,du \\ -\int_{c_2^{-1}(p_{\text{II}}-c_p)}^{z_{\text{II}}}(c_2(u) - h_1)g(u)\,du - (c_2(z_{\text{II}}) - h_1)z_{\text{II}}g(z_{\text{II}})\end{array}\right] \\ &= \left(\frac{\partial Den(p_{\text{II}}, z_{\text{II}})}{\partial z_{\text{II}}}\right) - (c_2(z_{\text{II}}) - h_1)z_{\text{II}}g(z_{\text{II}}) \end{aligned}$$

$$Den(p_{\mathrm{II}}, z_{\mathrm{II}}) = \left[\begin{array}{c} (p_{\mathrm{II}} - c_p - h_1) z_{\mathrm{II}} - \int_0^{c_2^{-1}(p_{\mathrm{II}} - c_p)} (p_{\mathrm{II}} - c_p - h_1)(z_{\mathrm{II}} - u) g(u) du \\ - \int_{c_2^{-1}(p_{\mathrm{II}} - c_p)}^{z_{\mathrm{II}}} (c_2(u) - h_1)(z_{\mathrm{II}} - u) g(u) du \end{array} \right]$$

$$\frac{\partial Den(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}} = \left[\begin{array}{c} (p_{\mathrm{II}} - c_p - h_1) - \int_0^{c_2^{-1}(p_{\mathrm{II}} - c_p)} (p_{\mathrm{II}} - c_p - h_1) g(u) du \\ - \int_{c_2^{-1}(p_{\mathrm{II}} - c_p)}^{z_{\mathrm{II}}} (c_2(u) - h_1) g(u) du \end{array} \right]$$

$$= (p_{\mathrm{II}} - c_p - h_1)\left(1 - G\left(c_2^{-1}(p_{\mathrm{II}} - c_p)\right)\right) - \int_{c_2^{-1}(p_{\mathrm{II}} - c_p)}^{z_{\mathrm{II}}} (c_2(u) - h_1) g(u) du > 0$$

$$\frac{\partial^2 Den(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}^2} = -(c_2(z_{\mathrm{II}}) - h_1) g(z_{\mathrm{II}}) < 0$$

$$\begin{aligned}
\frac{\partial Ratio(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}} &= \left(\frac{\partial Num(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right) \frac{1}{Den(p_{\mathrm{II}}, z_{\mathrm{II}})} - \left(\frac{\partial Den(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right) \frac{Ratio(p_{\mathrm{II}}, z_{\mathrm{II}})}{Den(p_{\mathrm{II}}, z_{\mathrm{II}})} \\
&= \left(\left(\frac{\partial Den(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right) - (c_2(z_{\mathrm{II}}) - h_1) z_{\mathrm{II}} g(z_{\mathrm{II}})\right) \frac{1}{Den(p_{\mathrm{II}}, z_{\mathrm{II}})} - \left(\frac{\partial Den(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right) \frac{Ratio(p_{\mathrm{II}}, z_{\mathrm{II}})}{Den(p_{\mathrm{II}}, z_{\mathrm{II}})} \\
&= \left(\left(\frac{Num(p_{\mathrm{II}}, z_{\mathrm{II}})}{z_{\mathrm{II}}}\right) - (c_2(z_{\mathrm{II}}) - h_1) z_{\mathrm{II}} g(z_{\mathrm{II}})\right) \frac{1}{Den(p_{\mathrm{II}}, z_{\mathrm{II}})} - \left(\frac{Num(p_{\mathrm{II}}, z_{\mathrm{II}})}{z_{\mathrm{II}}}\right) \frac{Ratio(p_{\mathrm{II}}, z_{\mathrm{II}})}{Den(p_{\mathrm{II}}, z_{\mathrm{II}})} \\
&= \left(\frac{Ratio(p_{\mathrm{II}}, z_{\mathrm{II}})}{z_{\mathrm{II}}}\right) \left[1 - \frac{(c_2(z_{\mathrm{II}}) - h_1) z_{\mathrm{II}}^2 g(z_{\mathrm{II}})}{Num(p_{\mathrm{II}}, z_{\mathrm{II}})} - Ratio(p_{\mathrm{II}}, z_{\mathrm{II}})\right] \\
&= \left(\frac{Ratio(p_{\mathrm{II}}, z_{\mathrm{II}})}{z_{\mathrm{II}}}\right) \left[1 - \frac{(c_2(z_{\mathrm{II}}) - h_1) z_{\mathrm{II}} g(z_{\mathrm{II}})}{\left(\frac{\partial Den(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right)} - Ratio(p_{\mathrm{II}}, z_{\mathrm{II}})\right] \\
&= \left(\frac{Ratio(p_{\mathrm{II}}, z_{\mathrm{II}})}{z_{\mathrm{II}}}\right) \left[1 + A(p_{\mathrm{II}}, z_{\mathrm{II}}) - Ratio(p_{\mathrm{II}}, z_{\mathrm{II}})\right]
\end{aligned}$$

where

$$A(p_{\mathrm{II}}, z_{\mathrm{II}}) = \frac{z\left(\frac{\partial^2 Den(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}^2}\right)}{\left(\frac{\partial Den(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right)} = -\frac{(c_2(z_{\mathrm{II}}) - h_1) z_{\mathrm{II}} g(z_{\mathrm{II}})}{\left(\frac{\partial Den(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right)} < 0$$

$$\begin{aligned}
\frac{\partial A(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}} &= \frac{\left(\frac{\partial^2 Den(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}^2}\right) + z_{\mathrm{II}}\left(-\left(\frac{\partial c_2(z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right) g(z_{\mathrm{II}}) - (c_2(z_{\mathrm{II}}) - h_1)\left(\frac{\partial g(z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right)\right) - \left(\frac{\partial^2 Den(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}^2}\right) A(p_{\mathrm{II}}, z_{\mathrm{II}})}{\left(\frac{\partial Den(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right)} \\
&= \frac{-(c_2(z_{\mathrm{II}}) - h_1)\left((1 - A(p_{\mathrm{II}}, z_{\mathrm{II}})) g(z_{\mathrm{II}}) + z_{\mathrm{II}}\left(\frac{\partial g(z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right)\right) - \left(\frac{\partial c_2(z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right) z_{\mathrm{II}} g(z_{\mathrm{II}})}{\left(\frac{\partial Den(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right)}
\end{aligned}$$

Note that we will show later $\frac{\partial A(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}} < 0$. From increasing generalized failure rate (IGFR), we know that

$$\begin{aligned}
r(z_{\mathrm{II}}) &= \frac{z_{\mathrm{II}} g(z_{\mathrm{II}})}{(1 - G(z_{\mathrm{II}}))} \\
\frac{\partial r(z_{\mathrm{II}})}{\partial z_{\mathrm{II}}} &= \frac{(1 + r(z_{\mathrm{II}})) g(z_{\mathrm{II}}) + z_{\mathrm{II}}\left(\frac{\partial g(z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right)}{(1 - G(z_{\mathrm{II}}))}
\end{aligned}$$

Next, observe that $(c_2(z_{\mathrm{II}}) - h_1)(1 - G(z_{\mathrm{II}})) < \left(\frac{\partial Den(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right)$.

$$\begin{aligned}
\frac{(c_2(z_{\mathrm{II}}) - h_1) z_{\mathrm{II}} g(z_{\mathrm{II}})}{(c_2(z_{\mathrm{II}}) - h_1)(1 - G(z_{\mathrm{II}}))} &> \frac{(c_2(z_{\mathrm{II}}) - h_1) z_{\mathrm{II}} g(z_{\mathrm{II}})}{\left(\frac{\partial Den(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right)} \\
r(z_{\mathrm{II}}) &> -A(p_{\mathrm{II}}, z_{\mathrm{II}}).
\end{aligned}$$

In order to show $\frac{\partial A(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}} < 0$, it is necessary to have

$$-\left(c_2\left(z_{\text{II}}\right)-h_1\right)\left(\left(1-A\left(p_{\text{II}},z_{\text{II}}\right)\right)g\left(z_{\text{II}}\right)+z_{\text{II}}\left(\frac{\partial g\left(z_{\text{II}}\right)}{\partial z_{\text{II}}}\right)\right)-\left(\frac{\partial c_2\left(z_{\text{II}}\right)}{\partial z_{\text{II}}}\right)z_{\text{II}}g\left(z_{\text{II}}\right)<0$$

$$\frac{-\left(\frac{\partial c_2(z_{\text{II}})}{\partial z_{\text{II}}}\right)}{\left(c_2\left(z_{\text{II}}\right)-h_1\right)}<\frac{\left(1-A\left(p_{\text{II}},z_{\text{II}}\right)\right)g\left(z_{\text{II}}\right)+\left(\frac{\partial g(z_{\text{II}})}{\partial z_{\text{II}}}\right)}{z_{\text{II}}g\left(z_{\text{II}}\right)}<\frac{\left(\frac{\partial r(z_{\text{II}})}{\partial z_{\text{II}}}\right)}{r\left(z_{\text{II}}\right)}$$

$$\frac{-z_{\text{II}}\left(\frac{\partial c_2(z_{\text{II}})}{\partial z_{\text{II}}}\right)}{\left(c_2\left(z_{\text{II}}\right)-h_1\right)}<\frac{z_{\text{II}}\left(\frac{\partial r(z_{\text{II}})}{\partial z_{\text{II}}}\right)}{r\left(z_{\text{II}}\right)}$$

Note that the above expression corresponds to the condition in (35). Therefore, when (35) is satisfied, we have $\frac{\partial A(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}} < 0$, and moreover,

$$\frac{\partial^2 Ratio\left(p_{\text{II}},z_{\text{II}}\right)}{\partial z_{\text{II}}^2}\mid_{\frac{\partial Ratio(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}=0}=\left(\frac{Ratio\left(p_{\text{II}},z_{\text{II}}\right)}{z_{\text{II}}}\right)\left(\frac{\partial A\left(p_{\text{II}},z_{\text{II}}\right)}{\partial z_{\text{II}}}\right)<0$$

The above result implies that once $Ratio\left(p_{\text{II}},z_{\text{II}}\right)$ starts decreasing, it remains decreasing. Therefore, it is sufficient to show that $Ratio\left(p_{\text{II}},z_{\text{II}}\right)$ is decreasing for $z_{\text{II}}=0$. First, note that $\lim_{z_{\text{II}}\to 0} Ratio\left(p_{\text{II}},z_{\text{II}}\right)=\frac{0}{0}$. Applying L'Hopital's rule:

$$\lim_{z_{\text{II}}\to 0} Ratio\left(p_{\text{II}},z_{\text{II}}\right)=\lim_{z_{\text{II}}\to 0}\frac{\left[\begin{array}{c}\left(p_{\text{II}}-c_p-h_1\right)-\int_0^{c_2^{-1}(p_{\text{II}}-c_p)}\left(p_{\text{II}}-c_p-h_1\right)g\left(u\right)du\\-\int_{c_2^{-1}(p_{\text{II}}-c_p)}^{z_{\text{II}}}\left(c_2\left(u\right)-h_1\right)g\left(u\right)du-\left(c_2\left(z_{\text{II}}\right)-h_1\right)z_{\text{II}}g\left(z_{\text{II}}\right)\end{array}\right]}{\left[\begin{array}{c}\left(p_{\text{II}}-c_p-h_1\right)-\int_0^{c_2^{-1}(p_{\text{II}}-c_p)}\left(p_{\text{II}}-c_p-h_1\right)g\left(u\right)du\\-\int_{c_2^{-1}(p_{\text{II}}-c_p)}^{z_{\text{II}}}\left(c_2\left(u\right)-h_1\right)g\left(u\right)du\end{array}\right]}=\frac{1}{1}.$$

Consider

$$\begin{aligned}\lim_{z_{\text{II}}\to 0}\left(\frac{\partial Ratio\left(p_{\text{II}},z_{\text{II}}\right)}{\partial z_{\text{II}}}\right) &= \lim_{z_{\text{II}}\to 0}\left[\frac{Den\left(p_{\text{II}},z_{\text{II}}\right)\left(\frac{\partial Num(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)-Num\left(p_{\text{II}},z_{\text{II}}\right)\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)}{\left(Den\left(p_{\text{II}},z_{\text{II}}\right)\right)^2}\right]\\ &\quad \lim_{z_{\text{II}}\to 0}\left[\frac{Den\left(p_{\text{II}},z_{\text{II}}\right)\left(\frac{\partial Num(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)-z_{\text{II}}\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)^2}{\left(Den\left(p_{\text{II}},z_{\text{II}}\right)\right)^2}\right]\\ &= \frac{0}{0}.\end{aligned}$$

Applying L'Hopital's rule:

$$\lim_{z_{\text{II}}\to 0}\left(\frac{\partial Ratio\left(p_{\text{II}},z_{\text{II}}\right)}{\partial z_{\text{II}}}\right)=\lim_{z_{\text{II}}\to 0}\left[\frac{\left[\begin{array}{c}\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)\left(\frac{\partial Num(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)+Den\left(p_{\text{II}},z_{\text{II}}\right)\left(\frac{\partial^2 Num(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}^2}\right)\\-\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)^2-2z_{\text{II}}\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)\left(\frac{\partial^2 Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}^2}\right)\end{array}\right]}{2Den\left(p_{\text{II}},z_{\text{II}}\right)\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)}\right]$$

$$=\lim_{z_{\text{II}}\to 0}\left[\frac{\left[\begin{array}{c}\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}-\left(c_2\left(z_{\text{II}}\right)-h_1\right)z_{\text{II}}g\left(z_{\text{II}}\right)\right)+Den\left(p_{\text{II}},z_{\text{II}}\right)\left(\frac{\partial^2 Num(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}^2}\right)\\-\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)^2-2z_{\text{II}}\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)\left(\frac{\partial^2 Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}^2}\right)\end{array}\right]}{2Den\left(p_{\text{II}},z_{\text{II}}\right)\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)}\right]$$

$$=\lim_{z_{\text{II}}\to 0}\left[\frac{\left[\begin{array}{c}-\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)\left(c_2\left(z_{\text{II}}\right)-h_1\right)z_{\text{II}}g\left(z_{\text{II}}\right)+Den\left(p_{\text{II}},z_{\text{II}}\right)\left(\frac{\partial^2 Num(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}^2}\right)\\+2z_{\text{II}}\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)\left(c_2\left(z_{\text{II}}\right)-h_1\right)g\left(z_{\text{II}}\right)\end{array}\right]}{2Den\left(p_{\text{II}},z_{\text{II}}\right)\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)}\right]$$

$$=\lim_{z_{\text{II}}\to 0}\left[\frac{\left[\left(\frac{Num(p_{\text{II}},z_{\text{II}})}{z_{\text{II}}}\right)\left(c_2\left(z_{\text{II}}\right)-h_1\right)z_{\text{II}}g\left(z_{\text{II}}\right)+Den\left(p_{\text{II}},z_{\text{II}}\right)\left(-2\left(c_2\left(z_{\text{II}}\right)-h_1\right)g\left(z_{\text{II}}\right)-\left(\frac{\partial((c_2(z_{\text{II}})-h_1)g(z_{\text{II}}))}{\partial z_{\text{II}}}\right)\right)\right]}{2Den\left(p_{\text{II}},z_{\text{II}}\right)\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)}\right]$$

$$=\lim_{z_{\text{II}}\to 0}\left[\frac{\left(Ratio\left(p_{\text{II}},z_{\text{II}}\right)\right)\left(c_2\left(z_{\text{II}}\right)-h_1\right)g\left(z_{\text{II}}\right)}{2\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)}+\frac{\left(-2\left(c_2\left(z_{\text{II}}\right)-h_1\right)g\left(z_{\text{II}}\right)\right)}{2\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)}-\frac{\left(\frac{\partial((c_2(z_{\text{II}})-h_1)g(z_{\text{II}}))}{\partial z_{\text{II}}}\right)}{2\left(\frac{Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)}\right]$$

$$=\lim_{z_{\text{II}}\to 0}\left[\frac{\left(c_2\left(z_{\text{II}}\right)-h_1\right)g\left(z_{\text{II}}\right)}{2\left(\frac{\partial Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)}\left(Ratio\left(p_{\text{II}},z_{\text{II}}\right)-2\right)-\frac{\left(\frac{\partial((c_2(z_{\text{II}})-h_1)g(z_{\text{II}}))}{\partial z_{\text{II}}}\right)}{2\left(\frac{Den(p_{\text{II}},z_{\text{II}})}{\partial z_{\text{II}}}\right)}\right]$$

$$=\frac{-\left(c_2\left(0\right)-h_1\right)g\left(0\right)}{2}-0\leq 0.$$

Thus, $R\left(p_{\text{II}},z_{\text{II}}\right)$ is decreasing in z_{II} for a given p_{II}.

We first provide the proof for the linear demand case where demand is defined as in (1). Consider the first-order derivative w.r.t. z_{II} where the optimal price expression is substituted, corresponding to $\frac{\partial E[P_E(p_{\text{II}}(z_{\text{II}}),z_{\text{II}})]}{\partial z_{\text{II}}}=\frac{D(p_{\text{II}}(z_{\text{II}}))}{z_{\text{II}}^2}\times R\left(p_{\text{II}}\left(z_{\text{II}}\right),z_{\text{II}}\right)$. Note that it is sufficient to show $R\left(p_{\text{II}}\left(z_{\text{II}}\right),z_{\text{II}}\right)$ is decreasing in z_{II} in order to prove that $R\left(p_{\text{II}}\left(z_{\text{II}}\right),z_{\text{II}}\right)=0$ provides a unique optimal solution. Next substitute the optimal price expression in (40) using the linear demand function into $R\left(p_{\text{II}}\left(z_{\text{II}}\right),z_{\text{II}}\right)$.

$$\begin{aligned}
Den\left(p_{\text{II}}\left(z_{\text{II}}\right),z_{\text{II}}\right) &= \frac{1}{2}\left[\begin{array}{c}\left(\frac{a}{b}-\left(c_p+h_1\right)\right)\left(z_{\text{II}}-\int_0^{c_2^{-1}(p_{\text{II}}(z_{\text{II}})-c_p)}\left(z_{\text{II}}-u\right)g\left(u\right)du\right)\\-\int_{c_2^{-1}(p_{\text{II}}(z_{\text{II}})-c_p)}^{z_{\text{II}}}\left(c_2\left(u\right)-h_1\right)\left(z_{\text{II}}-u\right)g\left(u\right)du+\left(c-h_1\overline{u}\right)\end{array}\right]\\
&\geq \frac{1}{2}\left[\left(\frac{a}{b}-\left(c_p+h_1\right)\right)\left(z_{\text{II}}-\int_0^{z_{\text{II}}}\left(z_{\text{II}}-u\right)g\left(u\right)du\right)+\left(c-h_1\overline{u}\right)\right]\geq 0.\\
\frac{\partial Den\left(p_{\text{II}}\left(z_{\text{II}}\right),z_{\text{II}}\right)}{\partial z_{\text{II}}} &= \frac{1}{2}\left[\begin{array}{c}\left(\frac{a}{b}-\left(c_p+h_1\right)\right)\left(1-\int_0^{c_2^{-1}(p_{\text{II}}(z_{\text{II}})-c_p)}g\left(u\right)du\right)\\-\int_{c_2^{-1}(p_{\text{II}}(z_{\text{II}})-c_p)}^{z_{\text{II}}}\left(c_2\left(u\right)-h_1\right)g\left(u\right)du\end{array}\right]\\
&\geq \frac{1}{2}\left[\left(\frac{a}{b}-\left(c_p+h_1\right)\right)\left(1-\int_0^{z_{\text{II}}}g\left(u\right)du\right)\right]\geq 0.
\end{aligned}$$

Note that

$$\begin{aligned}
R\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right) &= -\left[Den\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right)\right] \times \left[1 - Ratio\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right)\right] + \left(c - h_1 \overline{u}\right) \\
\frac{\partial R\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right)}{\partial z_{\mathrm{II}}} &= -\left(\frac{\partial Den\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right)}{\partial z_{\mathrm{II}}}\right)\left[1 - Ratio\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right)\right] \\
&\quad -\left[Den\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right)\right]\left[-\left(\frac{\partial Ratio\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right)}{\partial z_{\mathrm{II}}}\right)\right].
\end{aligned}$$

It was already shown earlier that $\left(\frac{\partial Ratio(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right) \leq 0$, therefore $\frac{\partial R(p_{\mathrm{II}}(z_{\mathrm{II}}), z_{\mathrm{II}})}{\partial z_{\mathrm{II}}} \leq 0$.

Next, we provide the proof when demand is iso-elastic as in (2). Substituting the optimal price expression in (41) into $R\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right)$, we get:

$$\begin{aligned}
Den\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right) &= \left(\frac{1}{b-1}\right)\left[\begin{array}{c} \left(\left(c_p + h_1\right)\right)\left(z_{\mathrm{II}} - \int_0^{c_2^{-1}\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right) - c_p\right)} \left(z_{\mathrm{II}} - u\right) g\left(u\right) du\right) \\ + \int_{c_2^{-1}\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right) - c_p\right)}^{z_{\mathrm{II}}} \left(c_2\left(u\right) - h_1\right)\left(z_{\mathrm{II}} - u\right) g\left(u\right) du + b\left(c - h_1 \overline{u}\right) \end{array}\right] \geq 0. \\
\frac{\partial Den\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right)}{\partial z_{\mathrm{II}}} &= \left(\frac{1}{b-1}\right)\left[\begin{array}{c} \left(\left(c_p + h_1\right)\right)\left(1 - \int_0^{c_2^{-1}\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right) - c_p\right)} g\left(u\right) du\right) \\ + \int_{c_2^{-1}\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right) - c_p\right)}^{z_{\mathrm{II}}} \left(c_2\left(u\right) - h_1\right) g\left(u\right) du \end{array}\right] \geq 0.
\end{aligned}$$

Note that

$$\begin{aligned}
R\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right) &= -\left[Den\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right)\right] \times \left[1 - Ratio\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right)\right] + \left(c - h_1 \overline{u}\right) \\
\frac{\partial R\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right)}{\partial z_{\mathrm{II}}} &= -\left(\frac{\partial Den\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right)}{\partial z_{\mathrm{II}}}\right)\left[1 - Ratio\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right)\right] \\
&\quad -\left[Den\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right)\right]\left[-\left(\frac{\partial Ratio\left(p_{\mathrm{II}}\left(z_{\mathrm{II}}\right), z_{\mathrm{II}}\right)}{\partial z_{\mathrm{II}}}\right)\right].
\end{aligned}$$

It was already shown earlier that $\left(\frac{\partial Ratio(p_{\mathrm{II}}, z_{\mathrm{II}})}{\partial z_{\mathrm{II}}}\right) \leq 0$, therefore $\frac{\partial R(p_{\mathrm{II}}(z_{\mathrm{II}}), z_{\mathrm{II}})}{\partial z_{\mathrm{II}}} \leq 0$.

Proof of Lemma 6:

a) At the sale price $p = c_p + c_2\left(u = 0\right)$, we have $c_2^{-1}\left(p - c_p\right) = 0$. From (31), it is known that $\left(p - c_p - h_1\right)\int_0^{z_{LS}^*(p)} ug\left(u\right) du = \left(c_2\left(u = 0\right) - h_1\right)\int_0^{z_{LS}^*(p)} ug\left(u\right) du$. Because $c_2\left(u = 0\right) > h_1$ by definition, and because $\left(c_2\left(u = 0\right) - h_1\right)\int_0^{z_{LS}^*(p)} ug\left(u\right) du = c - h_1\overline{u}$ according to (31), $z_{LS}^*\left(p\right)$ must be strictly positive. Thus, $z_{LS}^*\left(p\right) > c_2^{-1}\left(p - c_p\right) = 0$.

For parts b), c) and d), recall that $c_2^{-1}\left(p - c_p\right) = 1$ when the sale price is $p = c_p + c_2\left(u = 1\right)$. Consider the case when $p = c_p + \frac{c}{\overline{u}}$; from (31), the firm has $\left(\frac{c}{\overline{u}} - h_1\right)\int_0^{z_{LS}^*\left(p = c_p + \frac{c}{\overline{u}}\right)} ug\left(u\right) du = c - h_1\overline{u}$. At this price level, the equality can only be satisfied when $z_{LS}^*\left(p = c_p + \frac{c}{\overline{u}}\right) = 1$ because $\left(\frac{c}{\overline{u}} - h_1\right)\int_0^{z_{LS}^*\left(p = c_p + \frac{c}{\overline{u}}\right)} ug\left(u\right) du = \left(\frac{c}{\overline{u}} - h_1\right)\overline{u} = c - h_1\overline{u}$. Furthermore, when $p > c_p + \frac{c}{\overline{u}}$, from (31), the firm has $z_{LS}^*\left(p\right) < 1$. And, when $p < c_p + \frac{c}{\overline{u}}$, the firm has $z_{LS}^*\left(p\right) = 1$ because $\left(p - c_p - h_1\right)\int_0^1 ug\left(u\right) du < c - h_1\overline{u}$. Therefore, when $c_2\left(u = 1\right) > \frac{c}{\overline{u}}$, at the sale price $p = c_p + c_2\left(u = 1\right)$, (31) provides that $z_{LS}^*\left(p\right) < 1$, and thus, $c_2^{-1}\left(p - c_p\right) = 1 > z_{LS}^*\left(p\right)$.

c) Similarly, when $c_2\left(u = 1\right) < \frac{c}{\overline{u}}$, the firm the firm has $z_{LS}^*\left(p\right)$ at its maximum value of 1 because $\left(p - c_p - h_1\right)\int_0^1 ug\left(u\right) du < c - h_1\overline{u}$. Thus, $c_2^{-1}\left(p - c_p\right) = z_{LS}^*\left(p\right) = 1$.

d) When $c_2\left(u = 1\right) < \frac{c}{\overline{u}}$, at the sale price $p = c_p + \frac{c}{\overline{u}}$, the firm has $z_{LS}^*\left(p\right) = 1$ and because $c_2^{-1}\left(p - c_p\right) < 1$, we get the result $z_{LS}^*\left(p\right) = 1 > c_2^{-1}\left(p - c_p\right)$.

Proposition 7:

a) For a given p_{II}, when $z^*_{LS}(p_{\text{II}}) > c_2^{-1}(p_{\text{II}} - c_p)$, consider the lost sales variant, where the optimal stocking level satisfies equation (31):

$$(p_{\text{II}} - c_p - h_1) \int_0^{z^*_{LS}(p_{\text{II}})} ug(u)\, du = c - h_1 \overline{u}.$$

Because $p_{\text{II}} - c_p - h_1 > c_2(u) - h_1$ in the region $c_2^{-1}(p - c_p) \le u < z^*_{LS}(p_{\text{II}})$, equation (38) becomes:

$$(p_{\text{II}} - c_p - h_1) \int_0^{c_2^{-1}(p_{\text{II}} - c_p)} ug(u)\, du + \int_{c_2^{-1}(p_{\text{II}} - c_p)}^{z^*_{LS}(p_{\text{II}})} (c_2(u) - h_1)\, ug(u)\, du \le (p_{\text{II}} - c_p - h_1) \int_0^{z^*_{LS}(p_{\text{II}})} ug(u)\, du = c - h_1 \overline{u}$$

and the left hand side of the equation for (EP) is smaller for the same value of $z^*_{LS}(p_{\text{II}})$. In order for (38) to satisfy the equality to $c - h_1\overline{u}$, the firm must have $z^*_{LS}(p_{\text{II}}) \le z^*_{\text{II}}(p_{\text{II}})$. Next, for the same p_{II} consider the stocking level for the complete backlogging variant in equation (26):

$$\int_0^{z^*_{CB}} (c_2(u) - h_1)\, ug(u)\, du = c - h_1 \overline{u},$$

and in (EP), the optimal stocking level is:

$$(p_{\text{II}} - c_p - h_1) \int_0^{c_2^{-1}(p_{\text{II}} - c_p)} ug(u)\, du + \int_{c_2^{-1}(p_{\text{II}} - c_p)}^{z^*_{\text{II}}(p_{\text{II}})} (c_2(u) - h_1)\, ug(u)\, du = c - h_1 \overline{u}.$$

Note that $p_{\text{II}} - c_p - h_1 < c_2(u) - h_1$ in the region $0 \le u < c_2^{-1}(p_{\text{II}} - c_p)$, and when $z^*_{\text{II}}(p_{\text{II}}) > z^*_{LS}(p_{\text{II}}) > c_2^{-1}(p_{\text{II}} - c_p)$ if we were to assume $z^*_{\text{II}}(p_{\text{II}}) = z^*_{CB}$, the firm would have:

$$(p_{\text{II}} - c_p - h_1) \int_0^{c_2^{-1}(p_{\text{II}} - c_p)} ug(u)\, du + \int_{c_2^{-1}(p_{\text{II}} - c_p)}^{z^*_{CB}} (c_2(u) - h_1)\, ug(u)\, du < \int_0^{z^*_{CB}} (c_2(u) - h_1)\, ug(u)\, du = c - h_1 \overline{u}.$$

Therefore, the left hand side of the equation for (EP) must be smaller for the same value of z^*_{CB}. In order for (38) to satisfy the equality to $c - h_1\overline{u}$, the firm must have $z^*_{CB} < z^*_{\text{II}}(p_{\text{II}})$. Combining the two results for a given p_{II} that corresponds to $z^*_{LS}(p_{\text{II}}) > c_2^{-1}(p_{\text{II}} - c_p)$, the firm has $z^*_{\text{II}}(p_{\text{II}}) > z^*_{LS}(p_{\text{II}})$ and $z^*_{\text{II}}(p_{\text{II}}) > z^*_{CB}$.

b) When $z^*_{LS}(p_{\text{II}}) \le c_2^{-1}(p_{\text{II}} - c_p)$, the optimal stocking level in (EP) is:

$$\begin{pmatrix} (p_{\text{II}} - c_p - h_1) \int_0^{c_2^{-1}(p_{\text{II}} - c_p)} ug(u)\, du \\ + \int_{c_2^{-1}(p_{\text{II}} - c_p)}^{z^*_{\text{II}}(p_{\text{II}})} (c_2(u) - h_1)\, ug(u)\, du \end{pmatrix} = (p_{\text{II}} - c_p - h_1) \int_0^{z^*_{\text{II}}(p_{\text{II}})} ug(u)\, du = c - h_1 \overline{u}.$$

Because the above equation is equivalent to (31), $z^*_{\text{II}}(p_{\text{II}}) = z^*_{LS}(p_{\text{II}})$. Moreover, $p_{\text{II}} - c_p - h_1 < c_2(u) - h_1$ in the region $0 \le u \le z^*_{LS}(p_{\text{II}}) \le c_2^{-1}(p_{\text{II}} - c_p)$, therefore, $z^*_{\text{II}}(p_{\text{II}}) = z^*_{LS}(p_{\text{II}}) > z^*_{CB}$.

Proposition 8:

a) For a given $z_{\text{II}} < c_2^{-1}(p^*_{LS}(z_{\text{II}}) - c_p)$, the optimal sale price expression in (39) becomes the same equation with the one developed for the lost sales variant:

$$\begin{aligned} p^*_{\text{II}}(z_{\text{II}}) &= c_p + h_1 + \frac{\int_{c_2^{-1}(p_{\text{II}} - c_p)}^{z_{\text{II}}} (c_2(u) - h_1)(z_{\text{II}} - u)\, g(u)\, du}{\left(z_{\text{II}} - \int_0^{c_2^{-1}(p_{\text{II}} - c_p)} (z_{\text{II}} - u)\, g(u)\, du\right)} + \frac{c - h_1\overline{u}}{\left(z_{\text{II}} - \int_0^{c_2^{-1}(p_{\text{II}} - c_p)} (z_{\text{II}} - u)\, g(u)\, du\right)} - \frac{D(p_{\text{II}})}{\left(\frac{\partial D(p)}{\partial p}\,|_{p = p^*_{\text{II}}}\right)} \\ p^*_{\text{II}}(z_{\text{II}}) &= c_p + h_1 + \frac{c - h_1\overline{u}}{\left(z_{\text{II}} - \int_0^{z_{\text{II}}} (z_{\text{II}} - u)\, g(u)\, du\right)} - \frac{D(p_{\text{II}})}{\left(\frac{\partial D(p)}{\partial p}\,|_{p = p^*_{\text{II}}}\right)} \end{aligned}$$

Note that the above sale price expression is identical to the optimal sale price expression of the lost sales variant in (32).

b) For a given $z_{\mathrm{II}} > c_2^{-1}\left(p_{_{LS}}^*(z_{\mathrm{II}}) - c_p\right)$, from (31), the firm has:

$$\left(p_{LS}^*(z_{\mathrm{II}}) - c_p - h_1\right)\int_0^{z_{\mathrm{II}}} ug(u)\,du = c - h_1\overline{u}.$$

From (38), we have:

$$\left(p_{\mathrm{II}}(z_{\mathrm{II}}) - c_p - h_1\right)\int_0^{c_2^{-1}(p_{\mathrm{II}}-c_p)} ug(u)\,du + \int_{c_2^{-1}(p_{\mathrm{II}}-c_p)}^{z_{\mathrm{II}}} \left(c_2(u) - h_1\right) ug(u)\,du = c - h_1\overline{u}.$$

Note that if $p_{\mathrm{II}}(z_{\mathrm{II}})$ cannot be equal to $p_{LS}^*(z_{\mathrm{II}})$ because $c_2(u) - h_1 < p_{LS}^*(z_{\mathrm{II}}) - c_p - h_1$ in the region $c_2^{-1}(p_{LS}^* - c_p) < u \leq z_{\mathrm{II}}$, then if we assume $p_{\mathrm{II}}(z_{\mathrm{II}}) = p_{LS}^*(z_{\mathrm{II}})$ for the same z_{II}, equation (38) becomes

$$\begin{aligned}\left(\begin{array}{c}\left(p_{LS}^*(z_{\mathrm{II}}) - c_p - h_1\right)\int_0^{c_2^{-1}(p_{LS}^*(z_{\mathrm{II}})-c_p)} ug(u)\,du \\ +\int_{c_2^{-1}\left(p_{LS}^*(z_{\mathrm{II}})-c_p\right)}^{z_{\mathrm{II}}} \left(c_2(u) - h_1\right) ug(u)\,du\end{array}\right) &< \left(\begin{array}{c}\left(p_{LS}^*(z_{\mathrm{II}}) - c_p - h_1\right)\int_0^{c_2^{-1}(p_{LS}^*(z_{\mathrm{II}})-c_p)} ug(u)\,du \\ +\left(p_{LS}^*(z_{\mathrm{II}}) - c_p - h_1\right)\int_{c_2^{-1}\left(p_{LS}^*(z_{\mathrm{II}})-c_p\right)}^{z_{\mathrm{II}}} ug(u)\,du\end{array}\right) \\ &< \left(p_{LS}^*(z_{\mathrm{II}}) - c_p - h_1\right)\int_0^{z_{\mathrm{II}}} ug(u)\,du = c - h_1\overline{u}.\end{aligned}$$

Therefore, for a given $z_{\mathrm{II}} > c_2^{-1}(p_{LS}^*(z_{\mathrm{II}}) - c_p)$, $p_{\mathrm{II}}(z_{\mathrm{II}})$ must be greater than $p_{LS}^*(z_{\mathrm{II}})$ in order to satisfy the equality in (38).

Similarly, for a given $z_{\mathrm{II}} = z_{CB}^* > c_2^{-1}(p_{LS}^*(z_{\mathrm{II}}) - c_p)$ from (26) we have:

$$\int_0^{z_{\mathrm{II}}} \left[\left(c_2(u) - h_1\right) u\right] g(u)\,du = c - h_1\overline{u}.$$

From (38) we have:

$$\left(p_{\mathrm{II}}(z_{\mathrm{II}}) - c_p - h_1\right)\int_0^{c_2^{-1}(p_{\mathrm{II}}-c_p)} ug(u)\,du + \int_{c_2^{-1}(p_{\mathrm{II}}-c_p)}^{z_{\mathrm{II}}} \left(c_2(u) - h_1\right) ug(u)\,du = c - h_1\overline{u}.$$

Note that if $p_{\mathrm{II}}(z_{\mathrm{II}})$ cannot be equal to $p_{CB}^*(z_{\mathrm{II}})$ because $c_2(u) - h_1 > p_{LS}^*(z_{\mathrm{II}}) - c_p - h_1$ in the region $0 < u \leq c_2^{-1}(p_{LS}^*(z_{\mathrm{II}}) - c_p)$, then if we assume $p_{\mathrm{II}}(z_{_{\mathrm{II}}}) = p_{CB}^*(z_{\mathrm{II}})$ for the same z_{II}, equation (38) becomes

$$\begin{aligned}\left(\begin{array}{c}\left(p_{CB}^*(z_{\mathrm{II}}) - c_p - h_1\right)\int_0^{c_2^{-1}(p_{CB}^*(z_{\mathrm{II}})-c_p)} ug(u)\,du \\ +\int_{c_2^{-1}\left(p_{CB}^*(z_{\mathrm{II}})-c_p\right)}^{z_{\mathrm{II}}} \left(c_2(u) - h_1\right) ug(u)\,du\end{array}\right) &< \left(\begin{array}{c}\int_0^{c_2^{-1}(p_{CB}^*(z_{\mathrm{II}})-c_p)} \left(c_2(u) - h_1\right) ug(u)\,du \\ +\int_{c_2^{-1}\left(p_{CB}^*(z_{\mathrm{II}})-c_p\right)}^{z_{\mathrm{II}}} \left(c_2(u) - h_1\right) ug(u)\,du\end{array}\right) \\ &< \int_0^{z_{\mathrm{II}}} \left[\left(c_2(u) - h_1\right) u\right] g(u)\,du = c - h_1\overline{u}.\end{aligned}$$

Therefore, for a given $z_{\mathrm{II}} > c_2^{-1}(p_{LS}^*(z_{\mathrm{II}}) - c_p)$, $p_{\mathrm{II}}(z_{\mathrm{II}})$ must be less than $p_{CB}^*(z_{\mathrm{II}})$ in order to satisfy the equality in (38). Combining these two results leads to the fact that when $z_{\mathrm{II}} > c_2^{-1}(p_{LS}^*(z_{\mathrm{II}}) - c_p)$, the firm has $p_{LS}^*(z_{\mathrm{II}}) < p_{\mathrm{II}}^*(z_{\mathrm{II}}) < p_{CB}^*(z_{\mathrm{II}})$.

c) For a given $z_{\mathrm{II}} \geq z_{_{CB}}^*$, consider the optimality condition for the complete backlogging variant. From (27), the firm has:

$$p_{CB}^* = c_p + h_1 + \int_0^{z_{CB}^*} \left(c_2(u) - h_1\right) ug(u)\,du - \frac{D(p_{\mathrm{II}})}{\left(\frac{\partial D(p_{\mathrm{II}})}{\partial p_{\mathrm{II}}}\right)}.$$

First, consider the case when $z_{\text{II}} = z^*_{CB}$. From Cauchy-Schwarz inequality, it can be shown that:

$$\begin{aligned}
\left(\int_0^{z_{\text{II}}} (c_2(u) - h_1) g(u)\, du\right) \times \left(\int_0^{z_{\text{II}}} u g(u)\, du\right) &> \int_0^{z_{\text{II}}} (c_2(u) - h_1) u g(u)\, du \\
\int_0^{z_{\text{II}}} (c_2(u) - h_1) g(u)\, du &> \frac{\int_0^{z_{\text{II}}} (c_2(u) - h_1) u g(u)\, du}{\int_0^{z_{\text{II}}} u g(u)\, du} \\
\int_0^{z_{\text{II}}} (c_2(u) - h_1) g(u)\, du &> \frac{c - h_1}{\int_0^{z_{\text{II}}} u g(u)\, du} \\
\int_0^{z_{\text{II}}} (c_2(u) - h_1) g(u)\, du &> \left(\frac{c}{\overline{u}} - h_1\right)\left(\frac{\overline{u}}{\int_0^{z_{\text{II}}} u g(u)\, du}\right).
\end{aligned}$$

For the lost sales variant of the problem, the firm has to satisfy (32):

$$\begin{aligned}
p^*_{LS}(z_{\text{II}}) &= c_p + h_1 + \left(\frac{c}{\overline{u}} - h_1\right)\left(\frac{\overline{u}}{\left(z_{\text{II}} - \int_0^{z_{\text{II}}} (z_{\text{II}} - u) g(u)\, du\right)}\right) - \frac{D(p_{\text{II}})}{\left(\frac{\partial D(p_{\text{II}})}{\partial p_{\text{II}}}\right)} \\
p^*_{LS}(z_{\text{II}}) &= c_p + h_1 + \left(\frac{c}{\overline{u}} - h_1\right)\left(\frac{\overline{u}}{\left(z_{\text{II}}(1 - G(z_{\text{II}})) + \int_0^{z_{\text{II}}} u g(u)\, du\right)}\right) - \frac{D(p_{\text{II}})}{\left(\frac{\partial D(p_{\text{II}})}{\partial p_{\text{II}}}\right)}
\end{aligned}$$

Since $\frac{\overline{u}}{\int_0^{z_{\text{II}}} u g(u) du} \geq \frac{\overline{u}}{\left(z_{\text{II}}(1-G(z_{\text{II}}))+\int_0^{z_{\text{II}}} u g(u) du\right)}$ for all values of $z_{\text{II}} \geq z^*_{CB}$, then $p^*_{CB} > p^*_{LS}(z_{\text{II}})$.

Proof of Theorem 9:

a) The proof follows from Propositions 7 and 8. Proposition 7 a) states that $z^*_{\text{II}}(p_{\text{II}}) > z^*_{LS}(p_{\text{II}})$ and $z^*_{\text{II}}(p_{\text{II}}) > z^*_{CB}$ for any given p_{II}. It should be noted her that, according to Proposition 3, $z^*_{LS}(p_{\text{II}})$ is non-increasing in p_{II}. According to Theorem 2, it is known that z^*_{CB} does not depend on the sale price and is constant for the problem. Therefore, as the price level increases, $z^*_{LS}(p_{\text{II}})$ goes below z^*_{CB}. When $z^*_{LS}(p_{\text{II}}) \leq z^*_{CB}$ it can be seen that $z^*_{\text{II}}(p_{\text{II}}) > z^*_{CB}$, and therefore an LS policy cannot be optimal. Thus, the optimal policy can only be an LS policy when $z^*_{LS}(p_{\text{II}}) > z^*_{CB}$. In this condition, Proposition 7 b) states that when $z^*_{LS}(p_{\text{II}}) < c_2^{-1}(p_{\text{II}} - c_p)$ for the same p_{II}, then the firm would not benefit by increasing its stocking level. Therefore, it becomes $z^*_{\text{II}}(p_{\text{II}}) = z^*_{LS}(p_{\text{II}}) > z^*_{CB}$.

b) In the case when $z^*_{LS}(p_{\text{II}}) < c_2^{-1}(p_{\text{II}} - c_p)$, even if $z^*_{LS}(p_{\text{II}}) > z^*_{CB}$, the optimal policy in region II is a Combination policy because Proposition 7 a) states that the firm increases its objective function value by increasing its stocking level beyond $z^*_{LS}(p_{\text{II}})$. Therefore, $z^*_{\text{II}}(p_{\text{II}}) > z^*_{LS}(p_{\text{II}}) > z^*_{CB}$. It should be noted here that $z^*_{\text{II}}(p_{\text{II}}) > z^*_{LS}(p_{\text{II}})$ and $z^*_{\text{II}}(p_{\text{II}}) > z^*_{CB}$ for any given p_{II}. From Proposition 8 b) and 8 c) it is known that the firm has the optimal price expressions as $p^*_{LS}(z_{\text{II}}) < p^*_{\text{II}}(z_{\text{II}}) < p^*_{CB}$ for the stocking level values that are greater than z^*_{CB}. As a result, the firm has $p^*_{LS} \leq p^*_{\text{II}} \leq p^*_{CB}$.

Proof of Proposition 10:

In the region $p_{\text{III}} \geq c_p + c_2(u = 0)$, the firm would always engage in the purchasing option whenever $Qu < D(p)$ at every realization of u. Thus, the problem converges to the complete backlogging variant in this price range. This policy can be perceived as the optimal solution to the problem (EP) with the price constraint $p \geq c_p + c_2(u = 0)$.

a) Comparing (26) and (31), which can also be expressed as $\int_0^{z^*_{LS}} (p_{LS} - c_p - h_1) u g(u)\, du = c - h_1\overline{u}$, provides the result. Both stocking level expressions have the same right-hand side value of $(c - h_1\overline{u})$. Because $c_2(u) - h_1 \leq p_{\text{III}} - c_p - h_1$ for each realization of u, the stocking level that satisfies the right-hand side of (26) has to be greater than or equal to (31); thus, $z_{CB}^* > z_{LS}^*(p_{\text{III}})$ for any price $p_{\text{III}} \geq c_p + c_2(u=0)$. Therefore, if the optimal price for the problem is in the same range, then $z_{\text{III}}^* = z_{CB}^*$. However, if the optimal sale price is less than $c_p + c_2(u=0)$, then we already know that $z_{LS}(p)$ is non-increasing in p, and thus $z_{\text{III}}^*(p_{\text{III}}) = z_{CB}^* > z_{LS}(p_{\text{III}})$.

b) For any sale price $p_{\text{III}} \leq c_p + c_2(u=0)$, increasing q_2 increases the objective function by $p_{\text{III}} - c_p + c_2(u) > 0$ for each realization of u. Thus, satisfying the unmet demand through complete backlogging increases the objective function.

c) When $p_{\text{III}} \geq c_p + c_2(u=0)$, because of concavity, we already know that the maximum point is reached when $p_{\text{III}} = p_{CB}^*$ and $z_{\text{III}}^* = z_{CB}^* > z_{LS}(p_{\text{III}})$; therefore, $E[P_E(p_{\text{III}}^*, z_{\text{III}}^*)] = E\left[P_{E\_CB}(p_{CB}^*, z_{CB}^*)\right]$.

d) When $p_{\text{III}} < c_p + c_2(u=0)$, because of concavity, the maximum point is reached when $p_{\text{III}} = c_p + c_2(u=0)$. From Theorem 2, it is already known that the optimal stocking level can be determined independently of the sale price, and therefore, $z_{\text{III}}^* = z_{CB}^* > z_{LS}(p_{\text{III}})$. However, $E[P_E(p_{\text{III}}^*, z_{\text{III}}^*)] = E\left[P_{E\_CB}(p_{\text{III}}^*, z_{CB}^*)\right] \leq E\left[P_{E\_CB}(p_{CB}^*, z_{CB}^*)\right]$. Because there is already a better solution for the complete backlogging variant in region II, the optimal solution for (EP) cannot be in region III.

Proof of Theorem 11:

a) From Proposition 4.d) it is known that when $p_{LS}^* > c_p + c_2(u=1)$, the optimal solution cannot be located in region I. Proposition 10.c) states that the optimal solution cannot be in region III when $p_{CB}^* < c_p + c_2(u=0)$. Therefore, there is only possible region for the optimal solution, and it is located in region II. Thus, $p^* = p_{\text{II}}^*$ and $z^* = z_{\text{II}}^*$.

b) From Proposition 4.c), it is known that the optimal choice in region I is $p_{\text{I}}^* = p_{LS}^*$ and $z_{\text{I}}^* = z_{LS}^*$ when $p_{LS}^* \leq c_p + c_2(u=1)$. It is also known from Proposition 10.d) that the optimal solution cannot be located in region III when $p_{CB}^* < c_p + c_2(u=0)$. This leaves regions I and II as potentially optimal solutions. Thus, a comparison of the objective function values at $p_{\text{I}}^* = p_{LS}^*$ and $z_{\text{I}}^* = z_{LS}^*$ in region I and p_{II}^* and z_{II}^* of region II reveals the optimal solution for (EP).

c) From Proposition 4.d) it is known that when $p_{LS}^* > c_p + c_2(u=1)$, the optimal solution cannot be located in region I. Proposition 10.c) states that the optimal choice in region III is $p_{\text{III}}^* = p_{CB}^*$ and $z_{\text{III}}^* = z_{CB}^*$. This leaves regions II and III as potentially optimal solutions. Thus, a comparison of the objective function values at $p_{\text{III}}^* = p_{CB}^*$ and $z_{\text{III}}^* = z_{CB}^*$ in region III and p_{II}^* and z_{II}^* of region II reveals the optimal solution for (EP).

d) From Proposition 4.c) the optimal choice in region I is $p_{\text{I}}^* = p_{LS}^*$ and $z_{\text{I}}^* = z_{LS}^*$, and is a potentially optimal solution for for (EP). From Proposition 10.c) the optimal choice in region III is $p_{\text{III}}^* = p_{CB}^*$ and $z_{\text{III}}^* = z_{CB}^*$, and is also a potentially optimal solution for (EP). Thus, the optimal choices in each region can be potentially optimal for the original problem. Therefore, the optimal solution for (EP) can determined by comparing the objective function value of these three potentially optimal solutions: $p_{\text{I}}^* = p_{LS}^*$ and $z_{\text{I}}^* = z_{LS}^*$ of region I, p_{II}^* and z_{II}^* of region II, and $p_{\text{III}}^* = p_{CB}^*$ and $z_{\text{III}}^* = z_{CB}^*$ of region III.

e) From Proposition 4.d), the optimal solution cannot be in region I because $p_{LS}^* > c_p + c_2(u=0) > c_p + c_2(u=1)$. From Theorem 9 it is known that if the optimal solution were to be in region II, then the firm would have $p_{\text{II}}^* > p_{LS}^*$. However, because $p_{LS}^* > c_p + c_2(u=0) > p_{\text{II}}^*$ the condition in Theorem 9 is violated and therefore the optimal solution cannot be in region II either. Then, the two potentially optimal choices are p_{LS}^* and z_{LS}^* and

$p_{\text{III}}^* = p_{CB}^*$ and $z_{\text{III}}^* = z_{CB}^*$. From Lemma 10.a) it is known that $z_{\text{III}}^* = z_{CB}^* > z_{LS}^*$ when $p_{LS}^* > c_p + c_2(u = 0)$. Moreover, from Proposition 10.b), the firm has $E\left[P_{E\_CB}\left(p_{\text{III}} = p_{LS}^*, z_{CB}^*\right)\right] > E\left[P_{E\_LS}\left(p_{\text{III}} = p_{LS}^*, z_{LS}^*\left(p_{LS}^*\right)\right)\right]$. Therefore, $E\left[P_{E\_CB}\left(p_{CB}^*, z_{CB}^*\right)\right] > E\left[P_{E\_CB}\left(p_{\text{III}} = p_{LS}^*, z_{CB}^*\right)\right] > E\left[P_{E\_LS}\left(p_{\text{III}} = p_{LS}^*, z_{LS}^*\left(p_{LS}^*\right)\right)\right]$. As a result, the optimal solution is a CB policy with $p^* = p_{CB}^*$ and $z^* = z_{CB}^*$.

Proof of Corollary 12:

The optimal sale price is $p^* \geq p_{LS}^*$, and the firm has $D(p^*) \leq D(p_{LS}^*)$. It is also known that $z^* \geq z_{LS}^*$. Therefore, $z^* = \frac{D(p^*)}{Q^*} \geq z_{LS}^* = \frac{D(p_{LS}^*)}{Q_{LS}^*}$. Alternatively, $\frac{Q_{LS}^*}{Q^*} \geq \frac{D(p_{LS}^*)}{D(p^*)} \geq 1$; thus, $\frac{Q_{LS}^*}{Q^*} \geq 1$. As a result, $Q^* \leq Q_{LS}^*$.

Proof of Proposition 14:

We begin the proof by showing that $(q_1 + q_2) = D(p)$. When $(q_1 + q_2) < D(p)$, we have $\frac{\partial \pi(p,q_1,q_2|Q,u)}{\partial p} = (q_1 + q_2) > 0$, so increasing p, and thus decreasing the demand $D(p)$ to the level of $(q_1 + q_2)$, improves the objective function $\pi(p, q_1, q_2 \mid Q, u)$. Moreover, when $(q_1 + q_2) > D(p)$, we have:

1) $\frac{\partial \pi(p,q_1,q_2|Q,u)}{\partial q_1} = \left\{ \begin{smallmatrix} -c_p - h_1 + h_2 < 0 \text{ when } q_1 < Qu \\ -c_p + h_2 < 0 \text{ when } q_1 > Qu \end{smallmatrix} \right\}$, so decreasing q_1 to the level of $D(p)$ improves the objective function $\pi(p, q_1, q_2 \mid Q, u)$; and

2) $\frac{\partial \pi(p,q_1,q_2|Q,u)}{\partial q_2} = -c_p + c_2(u) < 0$, so decreasing q_2 to the level of $D(p)$ improves the objective function $\pi(p, q_1, q_2 \mid Q, u)$. Therefore, the optimal solution for the sum of production quantities is equal to the demand set by the optimal sale price, i.e., $(q_1^* + q_2^*) = D(p^*)$.

Consider the case when $(q_1 + q_2) < D(p)$, and observe that $\frac{\partial \pi(p,q_1,q_2|Q,u)}{\partial q_1} = \left\{ \begin{smallmatrix} p - c_p - h_1 \text{ when } q_1 \leq Qu \\ p - c_p \text{ when } q_1 > Qu \end{smallmatrix} \right\}$ and $\frac{\partial \pi(p,q_1,q_2|Q,u)}{\partial q_2} = p - c_p - c_2(u)$. For any given level of p, $\frac{\partial \pi(p,q_1,q_2|Q,u)}{\partial q_1} > \frac{\partial \pi(p,q_1,q_2|Q,u)}{\partial q_2}$. Therefore, when $q_1 \leq Qu$, the firm will increase its production through q_1 before it becomes necessary to increase q_2 to reach the level of the demand $D(p)$.

The first threshold point can be obtained by equating $q_1 = 0$ and utilizing q_2 to equate the amount of production to the demand, i.e., $q_2 = D(p)$.

$$\begin{aligned} \pi\left((p, q_1 = 0, q_2 = D(p) \mid Q, u)\right) &= (p - c_p - c_2(u))\, D(p) \\ \frac{\partial \pi\left((p, q_1 = 0, q_2 = D(p) \mid Q, u)\right)}{\partial p} &= (p - c_p - c_2(u)) \left(\frac{\partial D(p)}{\partial p}\right) + D(p) = 0 \\ p^* &= c_p + c_2(u) - \frac{D(p)}{\left(\frac{\partial D(p)}{\partial p}\right)}. \end{aligned}$$

When demand is defined by a a linear function as in (1), the optimal sale price expression becomes $p^* = \frac{a + b(c_p + c_2(u))}{2b}$, and when demand is iso-elastic as in (2), it becomes $p^* = \left(\frac{b}{b-1}\right)(c_p + c_2(u))$. Substituting these optimal sale price expressions into the demand function and equating it to q_2 provides the threshold point, $q_2^* = D(p^*)$. If demand is defined as in (1), then $q_2^* = D(p^*) = \frac{a - b(c_p + c_2(u))}{2}$; however, if demand is defined as in (2), then $q_2^* = D(p^*) = a\left(\left(\frac{b}{b-1}\right)(c_p + c_2(u))\right)^{-b}$. Note that when $0 < Qu < D(p)$ the firm's profit improves more by increasing q_1 compared to increasing q_2. Therefore, under linear demand as in (1), when $0 < Qu \leq \frac{a - b(c_p + c_2(u))}{2}$, the firm's optimal choices are:

$$p^* = \frac{a + b(c_p + c_2(u))}{2b}; q_1^* = Qu; q_2^* = \frac{a - b(c_p + c_2(u))}{2} - Qu.$$

Similarly, under iso-elastic demand as in (2), when $0 < Qu \leq a\left(\left(\frac{b}{b-1}\right)(c_p + c_2(u))\right)^{-b}$, the firm's optimal choices are:

$$p^* = \left(\frac{b}{b-1}\right)(c_p + c_2(u)); q_1^* = Qu; q_2^* = a\left(\left(\frac{b}{b-1}\right)(c_p + c_2(u))\right)^{-b} - Qu.$$

The second threshold point can be obtained by equating $q_2 = 0$. Consider the objective function when $Qu < D(p)$

$$\begin{aligned}
\pi((p, q_1, q_2 = 0 \mid Qu < D(p))) &= (p - c_p) q_1 + h_1 (Qu - q_1)^+ \\
\frac{\partial \pi((p, q_1, q_2 = 0 \mid Qu < D(p)))}{\partial p} &= q_1 \geq 0 \\
\frac{\partial \pi((p, q_1, q_2 = 0 \mid Qu < D(p)))}{\partial q_1} &= (p - c_p - h_1) \geq 0.
\end{aligned}$$

Thus, increasing p would reduce the demand $D(p)$, and the case $Qu < D(p)$ cannot happen. Next, consider the case when $Qu > D(p)$. Notice that increasing q_1 beyond $D(p)$ does not improve the objective function, so $q_1^* = D(p)$.

$$\begin{aligned}
\pi((p, q_1 = D(p), q_2 = 0 \mid Qu > D(p))) &= (p - c_p) D(p) + h_1 (Qu - D(p))^+ \\
\frac{\partial \pi((p, q_1, q_2 = 0 \mid Qu < D(p)))}{\partial p} &= (p - c_p - h_1)\left(\frac{\partial D(p)}{\partial p}\right) + D(p).
\end{aligned}$$

Equating $\frac{\partial \pi((p,q_1,q_2=0|Qu<D(p)))}{\partial p} = 0$ provides the optimal sale price at the threshold point. Using the linear demand function as in (1),

$$\begin{aligned}
p^* &= \frac{a + b(c_p + h_1)}{2b} \\
q_1^* &= D(p^*) = \frac{a - b(c_p + h_1)}{2}.
\end{aligned}$$

When demand is iso-elastic as in (2), then

$$\begin{aligned}
p^* &= \left(\frac{b}{b-1}\right)(c_p + h_1) \\
q_1^* &= D(p^*) = a\left(\left(\frac{b}{b-1}\right)(c_p + h_1)\right)^{-b}.
\end{aligned}$$

Using linear demand as in (1), when $Qu > \frac{a-b(c_p+h_1)}{2}$, the firm has the optimal choices as:

$$p^* = \frac{a + b(c_p + h_1)}{2b}; q_1^* = D(p^*) = \frac{a - b(c_p + h_1)}{2}; q_2^* = 0.$$

Using iso-elastic demand as in (2), when $Qu > a\left(\left(\frac{b}{b-1}\right)(c_p + h_1)\right)^{-b}$, the firm has the optimal choices as:

$$p^* = \left(\frac{b}{b-1}\right)(c_p + h_1); q_1^* = a\left(\left(\frac{b}{b-1}\right)(c_p + h_1)\right)^{-b}; q_2^* = 0.$$

Next, we investigate the optimal choices when the realized supply Qu is between these two threshold points. Since $q_2 = 0$ after the first threshold point, the objective function would improve when the firm has $q_1^* = D(p^*)$. Moreover, because the realized supply is less than or equal to the second threshold point, we also have $q_1^* = D(p^*) = Qu$. Thus, when the linear demand function as in (1) is used, and when $\frac{a-b(c_p+c_2(u))}{2} < Qu \leq \frac{a-b(c_p+h_1)}{2}$, the firm has $q_1^* = Qu = D(p^*) = a - bp^*$. The optimal choices are:

$$p^* = \frac{a - Qu}{b}; q_1^* = Qu; q_2^* = 0.$$

When the iso-elastic demand function as in (2) is used, and when $a\left(\left(\frac{b}{b-1}\right)(c_p+c_2(u))\right)^{-b} < Qu \leq a\left(\left(\frac{b}{b-1}\right)(c_p+h_1)\right)^{-b}$, the firm has $q_1^* = Qu = D(p^*) = a(p^*)^{-b}$. The optimal choices are:

$$p^* = \left(\frac{a}{Qu}\right)^{\frac{1}{b}}; q_1^* = Qu; q_2^* = 0.$$

Proof of Theorem 15:

Note that the objective function in (43) has two terms. The first term $-cQ$ is linear in Q, and it is sufficient to show that $PB(Q,u)$ is continuous and concave in Q as the integral preserves the concavity. For a given u, there are three regions of Qu that describes the objective function $\pi((p,q_1,q_2 \mid Q,u))$. Let us denote the objective function in each region as $\pi_i((p,q_1,q_2 \mid Q,u))$ where $i=1,2,3$ each one of these regions. We present the proof for the linear demand function as described by (1), and the same approach can be used in order to prove the same result under iso-elastic demand. When $0 \leq Qu \leq \frac{a-b(c_p+c_2(u))}{2}$, the objective function is:

$$\begin{aligned}
\pi_1((p,q_1,q_2 \mid Q,u)) &= \left(\frac{a+b(c_p+c_2(u))}{2b} - c_p - c_2(u)\right)\left(\frac{a-b(c_p+c_2(u))}{2}\right) + c_2(u)Qu \\
\frac{\partial \pi_1((p,q_1,q_2 \mid Q,u))}{\partial Q} &= c_2(u)u > 0 \\
\frac{\partial^2 \pi_1((p,q_1,q_2 \mid Q,u))}{\partial Q^2} &= 0.
\end{aligned}$$

Thus, the objective function $\pi_1((p,q_1,q_2 \mid Q,u))$ is linearly increasing in Q in the region $0 \leq Qu \leq \frac{a-b(c_p+c_2(u))}{2}$.

When $\frac{a-b(c_p+c_2(u))}{2} < Qu \leq \frac{a-b(c_p+h_1)}{2}$, the objective function is:

$$\begin{aligned}
\pi_2((p,q_1,q_2 \mid Q,u)) &= \left(\frac{a-Qu}{b} - c_p\right)Qu \\
\frac{\partial \pi_2((p,q_1,q_2 \mid Q,u))}{\partial Q} &= \left(\frac{a-2Qu}{b} - c_p\right)u \\
\frac{\partial^2 \pi_2((p,q_1,q_2 \mid Q,u))}{\partial Q^2} &= \frac{-2u^2}{b} \leq 0.
\end{aligned}$$

Thus, the objective function $\pi_2((p,q_1,q_2 \mid Q,u))$ is concave in Q in the region $\frac{a-b(c_p+c_2(u))}{2} < Qu \leq \frac{a-b(c_p+h_1)}{2}$.

When $Qu > \frac{a-b(c_p+h_1)}{2}$, the objective function is:

$$\begin{aligned}
\pi_3((p,q_1,q_2 \mid Q,u)) &= \left(\frac{a+b(c_p+h_1)}{2b} - c_p - h_1\right)\left(\frac{a-b(c_p+h_1)}{2}\right) + h_1 Qu \\
\frac{\partial \pi_3((p,q_1,q_2 \mid Q,u))}{\partial Q} &= h_1 u \\
\frac{\partial^2 \pi_3((p,q_1,q_2 \mid Q,u))}{\partial Q^2} &= 0.
\end{aligned}$$

Thus, the objective function $\pi_3((p,q_1,q_2 \mid Q,u))$ is linearly increasing in Q in the region $Qu > \frac{a-b(c_p+h_1)}{2}$. Thus, the objective function $\pi((p,q_1,q_2 \mid Q,u))$ is linear in Q in two regions and concave in Q in another; thus $\pi((p,q_1,q_2 \mid Q,u))$ is concave in Q.

It should be observed that $\pi_1((p,q_1,q_2 \mid Q,u)) = \pi_2((p,q_1,q_2 \mid Q,u))$ when $Qu = \frac{a-b(c_p+c_2(u))}{2}$ and $\pi_2((p,q_1,q_2 \mid Q,u)) = \pi_3((p,q_1,q_2 \mid Q,u))$ when $Qu = \frac{a-b(c_p+h_1)}{2}$, proving that $\pi((p,q_1,q_2 \mid Q,u))$ is continuous in Q. As a result, because $\pi((p,q_1,q_2 \mid Q,u))$ is continuous and concave in Q, $PB(Q,u)$ is also continuous and concave in Q. This completes the proof that $E[P_E(Q)]$ is continuous and concave in Q.