Online Appendix

for

The Impact of Process Deterioration on Production and Maintenance Policies

March 12, 2012

Proof of Proposition 1: We first derive the critical ratio expression by equating the expected values of the two policies $\mathbf{A}_9 = [P2, P1, P1, M1]$ and $\mathbf{A}_{25} = [P2, P1, M1, M1]$. Recall that

$$\widehat{\Pi}_1(\mathbf{A}_9) = \widehat{\Pi}_1(\mathbf{A}_{25}) - \Delta_{1,3}^{M1,P1} \tag{17}$$

$$\widehat{\Pi}_2(\mathbf{A}_9) = \widehat{\Pi}_2(\mathbf{A}_{25}) - \Delta_{2,3}^{M1,P1} \tag{18}$$

where $0 < \Delta_{1,3}^{M1,P1} < \widehat{\Pi}_1(\mathbf{A}_{25})$ and $0 < \Delta_{2,3}^{M1,P1} < \widehat{\Pi}_2(\mathbf{A}_{25})$ are the changes in the numerator terms of states 1 and 2 when the firm switches from implementing the maintenance action $M1$ to manufacturing product $P1$ in state 3. The steady-state probabilities of state 3 are not equal for these two policies, the numerator expressions are equal, i.e., $\widehat{\Pi}_3(\mathbf{A}_9) = \widehat{\Pi}_3(\mathbf{A}_{25})$. The worst state resembles the best state with some changes:

$$\widehat{\Pi}_4(\mathbf{A}_9) = \widehat{\Pi}_4(\mathbf{A}_{25}) + \Delta_{4,3}^{M1,P1} \tag{19}$$

where $\Delta_{4,3}^{M1,P1} > 0$.

$$\begin{aligned}
EV(\mathbf{A}_9 = [P2, P1, P1, M1]) &= EV(\mathbf{A}_{25} = [P2, P1, M1, M1]) \\
\frac{\begin{bmatrix} r_{1,P2}\Pi_1(\mathbf{A}_9) + r_{2,P1}\Pi_2(\mathbf{A}_9) \\ +r_{3,P1}\Pi_3(\mathbf{A}_9) - c_{4,M1}\Pi_4(\mathbf{A}_9) \end{bmatrix}}{\begin{bmatrix} \tau_{1,P2}\Pi_1(\mathbf{A}_9) + \tau_{2,P1}\Pi_2(\mathbf{A}_9) \\ +\tau_{3,P1}\Pi_3(\mathbf{A}_9) + \tau_{4,M1}\Pi_4(\mathbf{A}_9) \end{bmatrix}} &= \frac{\begin{bmatrix} r_{1,P2}\Pi_1(\mathbf{A}_{25}) + r_{2,P1}\Pi_2(\mathbf{A}_{25}) \\ -c_{3,M1}\Pi_3(\mathbf{A}_{25}) - c_{4,M1}\Pi_4(\mathbf{A}_{25}) \end{bmatrix}}{\begin{bmatrix} \tau_{1,P2}\Pi_1(\mathbf{A}_{25}) + \tau_{2,P1}\Pi_2(\mathbf{A}_{25}) \\ +\tau_{3,M1}\Pi_3(\mathbf{A}_{25}) + \tau_{4,M1}\Pi_4(\mathbf{A}_{25}) \end{bmatrix}} \\
\frac{\begin{bmatrix} r_{1,P2}\widehat{\Pi}_1(\mathbf{A}_9) + r_{2,P1}\widehat{\Pi}_2(\mathbf{A}_9) \\ +r_{3,P1}\widehat{\Pi}_3(\mathbf{A}_9) - c_{4,M1}\widehat{\Pi}_4(\mathbf{A}_9) \end{bmatrix}}{\begin{bmatrix} \tau_{1,P2}\widehat{\Pi}_1(\mathbf{A}_9) + \tau_{2,P1}\widehat{\Pi}_2(\mathbf{A}_9) \\ +\tau_{3,P1}\widehat{\Pi}_3(\mathbf{A}_9) + \tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_9) \end{bmatrix}} &= \frac{\begin{bmatrix} r_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25}) + r_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25}) \\ -c_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25}) - c_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25}) \end{bmatrix}}{\begin{bmatrix} \tau_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25}) + \tau_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25}) \\ +\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25}) + \tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25}) \end{bmatrix}}
\end{aligned}$$

Substituting (17), (18) and (19) into $EV(\mathbf{A}_9 = [P2, P1, P1, M1])$ provides:

$$\frac{r_{1,P2}\left(\widehat{\Pi}_1(\mathbf{A}_{25}) - \Delta_{1,3}^{M1,P1}\right) + r_{2,P1}\left(\widehat{\Pi}_2(\mathbf{A}_{25}) - \Delta_{2,3}^{M1,P1}\right) + r_{3,P1}\widehat{\Pi}_3(\mathbf{A}_{25}) - c_{4,M1}\left(\widehat{\Pi}_4(\mathbf{A}_{25}) + \Delta_{4,3}^{M1,P1}\right)}{\tau_{1,P2}\left(\widehat{\Pi}_1(\mathbf{A}_{25}) - \Delta_{1,3}^{M1,P1}\right) + \tau_{2,P1}\left(\widehat{\Pi}_2(\mathbf{A}_{25}) - \Delta_{2,3}^{M1,P1}\right) + \tau_{3,P1}\widehat{\Pi}_3(\mathbf{A}_{25}) + \tau_{4,M1}\left(\widehat{\Pi}_4(\mathbf{A}_{25}) + \Delta_{4,3}^{M1,P1}\right)} =$$

$$\frac{r_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25}) + r_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25}) - c_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25}) - c_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25})}{\tau_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25}) + \tau_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25}) + \tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25}) + \tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25})}$$

By adding and subtracting the terms $\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25})$ in the denominator of $EV(\mathbf{A}_9 = [P2, P1, P1, M1])$ provides:

$$\begin{aligned}
& r_{1,P2}\left(\widehat{\Pi}_1(\mathbf{A}_{25}) - \Delta_{1,3}^{M1,P1}\right) + r_{2,P1}\left(\widehat{\Pi}_2(\mathbf{A}_{25}) - \Delta_{2,3}^{M1,P1}\right) + r_{3,P1}\widehat{\Pi}_3(\mathbf{A}_{25}) - c_{4,M1}\left(\widehat{\Pi}_4(\mathbf{A}_{25}) + \Delta_{4,3}^{M1,P1}\right) \\
= \;& r_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25}) + r_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25}) - c_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25}) - c_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25}) \\
& +EV(\mathbf{A}_{25})\left\{-\tau_{1,P2}\Delta_{1,3}^{M1,P1} - \tau_{2,P1}\Delta_{2,3}^{M1,P1} + (\tau_{3,P1} - \tau_{3,M1})\widehat{\Pi}_3(\mathbf{A}_{25}) + \tau_{4,M1}\Delta_{4,3}^{M1,P1}\right\}
\end{aligned}$$

$$r_{3,P1}\widehat{\Pi}_3(\mathbf{A}_{25}) = -c_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25}) + r_{1,P2}\Delta_{1,3}^{M1,P1} + r_{2,P1}\Delta_{2,3}^{M1,P1} + c_{4,M1}\Delta_{4,3}^{M1,P1}$$
$$+EV(\mathbf{A}_{25})\left\{-\tau_{1,P2}\Delta_{1,3}^{M1,P1} - \tau_{2,P1}\Delta_{2,3}^{M1,P1} + (\tau_{3,P1}-\tau_{3,M1})\widehat{\Pi}_3(\mathbf{A}_{25}) + \tau_{4,M1}\Delta_{4,3}^{M1,P1}\right\}$$

$$r_{2,P1} = -c_{3,M1} + r_{1,P2}\left(\frac{\Delta_{1,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{25})}\right) + r_{2,P1}\left(\frac{\Delta_{2,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{25})}\right) + c_{4,M1}\left(\frac{\Delta_{4,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{25})}\right)$$
$$+EV(\mathbf{A}_{25})\left\{-\tau_{1,P1}\left(\frac{\Delta_{1,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{25})}\right) - \tau_{2,P1}\left(\frac{\Delta_{2,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{25})}\right) + (\tau_{3,P1}-\tau_{3,M1}) + \tau_{4,M1}\left(\frac{\Delta_{4,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{25})}\right)\right\}$$

$$\frac{r_{3,P1}}{c_{3,M1}} = -1 + \left(\frac{r_{1,P1}}{c_{3,M1}}\right)\left(\frac{\Delta_{1,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{25})}\right) + \left(\frac{r_{2,P1}}{c_{3,M1}}\right)\left(\frac{\Delta_{2,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{25})}\right) + \left(\frac{c_{4,M1}}{c_{3,M1}}\right)\left(\frac{\Delta_{4,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{25})}\right)$$
$$+\left(\frac{EV(\mathbf{A}_{25})}{c_{3,M1}}\right)\left\{-\tau_{1,P1}\left(\frac{\Delta_{1,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{25})}\right) - \tau_{2,P1}\left(\frac{\Delta_{2,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{25})}\right) + (\tau_{3,P1}-\tau_{3,M1}) + \tau_{4,M1}\left(\frac{\Delta_{4,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{25})}\right)\right\}$$

$$\frac{r_{3,P1}}{c_{3,M1}} = -1 + \left(\frac{1}{c_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25})}\right)\left[\begin{array}{c} r_{1,P2}\Delta_{1,3}^{M1,P1} + r_{2,P1}\Delta_{2,3}^{M1,P1} + c_{4,M1}\Delta_{4,3}^{M1,P1} \\ +EV(\mathbf{A}_{25})\left\{\begin{array}{c} -\tau_{1,P2}\Delta_{1,3}^{M1,P1} - \tau_{2,P1}\Delta_{2,3}^{M1,P1} \\ +(\tau_{3,P1}-\tau_{3,M1})\widehat{\Pi}_3(\mathbf{A}_{25}) + \tau_{4,M1}\Delta_{4,3}^{M1,P1} \end{array}\right\} \end{array}\right]$$

Note that $EV(\mathbf{A}_{25})\times\left(-\tau_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25}) - \tau_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25}) - \tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25}) - \tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25})\right) = -r_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25}) - r_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25}) + c_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25}) + c_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25})$. Therefore, substituting $-\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25})EV(\mathbf{A}_{25}) = -r_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25}) - r_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25}) + c_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25}) + c_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25}) + EV(\mathbf{A}_{25})\times\left(\tau_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25})\,\tau_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25})\ +\tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25})\right)$ into the above expression provides

$$\frac{r_{3,P1}}{c_{3,M1}} = -1 + \left(\frac{1}{c_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25})}\right)\left[\begin{array}{c} -r_{1,P2}\left(\widehat{\Pi}_1(\mathbf{A}_{25}) - \Delta_{1,3}^{M1,P1}\right) - r_{2,P1}\left(\widehat{\Pi}_2(\mathbf{A}_{25}) - \Delta_{2,3}^{M1,P1}\right) \\ +c_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25}) + c_{4,M1}\left(\widehat{\Pi}_4(\mathbf{A}_{25}) + \Delta_{4,3}^{M1,P1}\right) \\ +EV(\mathbf{A}_{25})\left\{\begin{array}{c} \tau_{1,P2}\left(\widehat{\Pi}_1(\mathbf{A}_{25}) - \Delta_{1,3}^{M1,P1}\right) \\ \tau_{2,P1}\left(\widehat{\Pi}_2(\mathbf{A}_{25}) - \Delta_{2,3}^{M1,P1}\right) \\ +\tau_{3,P1}\widehat{\Pi}_3(\mathbf{A}_{25}) \\ +\tau_{4,M1}\left(\widehat{\Pi}_4(\mathbf{A}_{25}) + \Delta_{4,3}^{M1,P1}\right) \end{array}\right\} \end{array}\right]$$

$$\frac{r_{3,P1}}{c_{3,M1}} = -\left(\frac{r_{1,P2}}{c_{3,M1}}\right)\left(\frac{\widehat{\Pi}_1(\mathbf{A}_{25}) - \Delta_{1,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{25})}\right) - \left(\frac{r_{2,P1}}{c_{3,M1}}\right)\left(\frac{\widehat{\Pi}_2(\mathbf{A}_{25}) - \Delta_{2,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{25})}\right)$$
$$+\left(\frac{c_{4,M1}}{c_{3,M1}}\right)\left(\frac{\left(\widehat{\Pi}_4(\mathbf{A}_{25}) + \Delta_{4,3}^{M1,P1}\right)}{\widehat{\Pi}_3(\mathbf{A}_{25})}\right) + EV(\mathbf{A}_{25})\left\{\begin{array}{c} \tau_{1,P2}\left(\widehat{\Pi}_1(\mathbf{A}_{25}) - \Delta_{1,3}^{M1,P1}\right) \\ \tau_{2,P1}\left(\widehat{\Pi}_2(\mathbf{A}_{25}) - \Delta_{2,3}^{M1,P1}\right) \\ +\tau_{3,P1}\widehat{\Pi}_3(\mathbf{A}_{25}) \\ +\tau_{4,M1}\left(\widehat{\Pi}_4(\mathbf{A}_{25}) + \Delta_{4,3}^{M1,P1}\right) \end{array}\right\}$$

The critical ratio $\gamma_3^{M1,P1}$ for switching from minor maintenance $M1$ to low-end technology product $P1$ is then expressed as follows:

$$\gamma_3^{M1,P1} = \frac{r_{3,P1}}{c_{3,M1}} = -\left(\frac{r_{1,P2}}{c_{3,M1}}\right)\left(\frac{\widehat{\Pi}_1(\mathbf{A}_{25}) - \Delta_{1,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{25})}\right) - \left(\frac{r_{2,P1}}{c_{3,M1}}\right)\left(\frac{\widehat{\Pi}_2(\mathbf{A}_{25}) - \Delta_{2,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{25})}\right)$$
$$+\left(\frac{c_{4,M1}}{c_{3,M1}}\right)\left(\frac{\left(\widehat{\Pi}_4(\mathbf{A}_{25}) + \Delta_{4,3}^{M1,P1}\right)}{\widehat{\Pi}_2(\mathbf{A}_{25})}\right) + EV(\mathbf{A}_{25})\left\{\begin{array}{c} \tau_{1,P2}\left(\widehat{\Pi}_1(\mathbf{A}_{25}) - \Delta_{1,3}^{M1,P1}\right) \\ \tau_{2,P1}\left(\widehat{\Pi}_2(\mathbf{A}_{25}) - \Delta_{2,3}^{M1,P1}\right) \\ +\tau_{3,P1}\widehat{\Pi}_3(\mathbf{A}_{25}) \\ +\tau_{4,M1}\left(\widehat{\Pi}_4(\mathbf{A}_{25}) + \Delta_{4,3}^{M1,P1}\right) \end{array}\right\}.$$

Therefore, when $\frac{r_{3,P1}}{c_{3,M1}} = \gamma_3^{M1,P1}$, then the firm is indifferent between production and maintenance actions in state 3 because $EV(\mathbf{A}_9 = [P2, P1, P1, M1]) = EV(\mathbf{A}_{25} = [P2, P1, M1, M1])$. This proves part c) of the Proposition. Note that the value of $\frac{r_{3,P1}}{c_{3,M1}}$ is positive by definition, and therefore, if $\gamma_3^{M1,P1} < 0$, it implies that $EV(\mathbf{A}_9 = [P2, P1, P1, M1]) > EV(\mathbf{A}_{25} = [P2, P1, M1, M1])$, and therefore $a_3^* = P1$. a) When $\frac{r_{3,P1}}{c_{3,M1}} > \gamma_3^{M1,P1}$, the firm has $EV(\mathbf{A}_9 = [P2, P1, P1, M1]) > EV(\mathbf{A}_{25} = [P2, P1, M1, M1])$, and therefore $a_3^* = P1$. b) When $\frac{r_{3,P1}}{c_{3,M1}} < \gamma_3^{M1,P1}$, the firm has $EV(\mathbf{A}_9 = [P2, P1, P1, M1]) < EV(\mathbf{A}_{25} = [P2, P1, M1, M1])$ and therefore $a_3^* = M1$.

Proof of Proposition 2: We first derive the critical ratio expression by equating the expected values of the two policies $\mathbf{A}_9 = [P2, P1, P1, M1]$ and $\mathbf{A}_{11} = [P2, P1, P2, M1]$. Recall that

$$\widehat{\Pi}_1(\mathbf{A}_{11}) = \widehat{\Pi}_1(\mathbf{A}_9) \times \delta_3^{P1,P2}, \quad (20)$$
$$\widehat{\Pi}_2(\mathbf{A}_{11}) = \widehat{\Pi}_2(\mathbf{A}_9) \times \delta_3^{P1,P2}, \quad (21)$$
$$\widehat{\Pi}_3(\mathbf{A}_{11}) = \widehat{\Pi}_3(\mathbf{A}_9), \quad (22)$$
$$\widehat{\Pi}_4(\mathbf{A}_{11}) = \widehat{\Pi}_4(\mathbf{A}_9) \times \delta_3^{P1,P2}. \quad (23)$$

Using these relationships, we can express these two policies as follows:

$$EV(\mathbf{A}_{11} = [P2, P1, P2, M1]) = EV(\mathbf{A}_9 = [P2, P1, P1, M1])$$
$$\frac{\left[\begin{array}{c} r_{1,P2}\Pi_1(\mathbf{A}_{11}) + r_{2,P1}\Pi_2(\mathbf{A}_{11}) \\ +r_{3,P2}\Pi_3(\mathbf{A}_{11}) - c_{4,M1}\Pi_4(\mathbf{A}_{11}) \end{array}\right]}{\left[\begin{array}{c} \tau_{1,P2}\Pi_1(\mathbf{A}_{11}) + \tau_{2,P1}\Pi_2(\mathbf{A}_{11}) \\ +\tau_{3,P2}\Pi_3(\mathbf{A}_{11}) + \tau_{4,M1}\Pi_4(\mathbf{A}_{11}) \end{array}\right]} = \frac{\left[\begin{array}{c} r_{1,P2}\Pi_1(\mathbf{A}_9) + r_{2,P1}\Pi_2(\mathbf{A}_9) \\ +r_{3,P1}\Pi_3(\mathbf{A}_9) - c_{4,M1}\Pi_4(\mathbf{A}_9) \end{array}\right]}{\left[\begin{array}{c} \tau_{1,P2}\Pi_1(\mathbf{A}_9) + \tau_{2,P1}\Pi_2(\mathbf{A}_9) \\ +\tau_{3,P1}\Pi_3(\mathbf{A}_9) + \tau_{4,M1}\Pi_4(\mathbf{A}_9) \end{array}\right]}$$
$$\frac{\left[\begin{array}{c} r_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{11}) + r_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{11}) \\ +r_{3,P2}\widehat{\Pi}_3(\mathbf{A}_{11}) - c_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{11}) \end{array}\right]}{\left[\begin{array}{c} \tau_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{11}) + \tau_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{11}) \\ +\tau_{3,P2}\widehat{\Pi}_3(\mathbf{A}_{11}) + \tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{11}) \end{array}\right]} = \frac{\left[\begin{array}{c} r_{1,P2}\widehat{\Pi}_1(\mathbf{A}_9) + r_{2,P1}\widehat{\Pi}_2(\mathbf{A}_9) \\ +r_{3,P1}\widehat{\Pi}_3(\mathbf{A}_9) - c_{4,M1}\widehat{\Pi}_4(\mathbf{A}_9) \end{array}\right]}{\left[\begin{array}{c} \tau_{1,P2}\widehat{\Pi}_1(\mathbf{A}_9) + \tau_{2,P1}\widehat{\Pi}_2(\mathbf{A}_9) \\ +\tau_{3,P1}\widehat{\Pi}_3(\mathbf{A}_9) + \tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_9) \end{array}\right]}$$

Substituting (20)–(23) into $EV(\mathbf{A}_{11} = [P2, P1, P2, M1])$ provides the following:

$$\frac{r_{1,P2}\left(\widehat{\Pi}_1(\mathbf{A}_9)\delta_3^{P1,P2}\right) + r_{2,P1}\left(\widehat{\Pi}_2(\mathbf{A}_9)\delta_3^{P1,P2}\right) + r_{3,P2}\widehat{\Pi}_3(\mathbf{A}_9) - c_{4,M1}\left(\widehat{\Pi}_4(\mathbf{A}_9)\delta_3^{P1,P2}\right)}{\tau_{1,P2}\left(\widehat{\Pi}_1(\mathbf{A}_9)\delta_3^{P1,P2}\right) + \tau_{2,P1}\left(\widehat{\Pi}_2(\mathbf{A}_9)\delta_3^{P1,P2}\right) + \tau_{3,P2}\widehat{\Pi}_3(\mathbf{A}_9) + \tau_{4,M1}\left(\widehat{\Pi}_4(\mathbf{A}_9)\delta_3^{P1,P2}\right)}$$
$$= \frac{r_{1,P2}\widehat{\Pi}_1(\mathbf{A}_9) + r_{2,P1}\widehat{\Pi}_2(\mathbf{A}_9) + r_{3,P1}\widehat{\Pi}_3(\mathbf{A}_9) - c_{4,M1}\widehat{\Pi}_4(\mathbf{A}_9)}{\tau_{1,P2}\widehat{\Pi}_1(\mathbf{A}_9) + \tau_{2,P1}\widehat{\Pi}_2(\mathbf{A}_9) + \tau_{3,P1}\widehat{\Pi}_3(\mathbf{A}_9) + \tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_9)}$$

Adding and subtracting $\tau_{3,P1}\widehat{\Pi}_3(\mathbf{A}_9)\,\delta_3^{P1,P2}$ into the denominator of $EV(\mathbf{A}_{11}=[P2,P1,P2,M1])$ provides:

$$\begin{aligned}
& r_{1,P2}\left(\widehat{\Pi}_1(\mathbf{A}_9)\,\delta_3^{P1,P2}\right)+r_{2,P1}\left(\widehat{\Pi}_2(\mathbf{A}_9)\,\delta_3^{P1,P2}\right)+r_{3,P2}\widehat{\Pi}_3(\mathbf{A}_9)-c_{4,M1}\left(\widehat{\Pi}_4(\mathbf{A}_9)\,\delta_3^{P1,P2}\right)\\
= & \; r_{1,P2}\left(\widehat{\Pi}_1(\mathbf{A}_9)\,\delta_3^{P1,P2}\right)+r_{2,P1}\left(\widehat{\Pi}_2(\mathbf{A}_9)\,\delta_3^{P1,P2}\right)+r_{3,P1}\left(\widehat{\Pi}_3(\mathbf{A}_9)\,\delta_3^{P1,P2}\right)-c_{4,M1}\left(\widehat{\Pi}_4(\mathbf{A}_9)\,\delta_3^{P1,P2}\right)\\
& +EV(\mathbf{A}_9)\,\widehat{\Pi}_3(\mathbf{A}_9)\left\{\tau_{3,P2}-\tau_{3,P1}\delta_3^{P1,P2}\right\}
\end{aligned}$$

$$r_{3,P2}\widehat{\Pi}_3(\mathbf{A}_9)=r_{3,P1}\left(\widehat{\Pi}_3(\mathbf{A}_9)\,\delta_3^{P1,P2}\right)+EV(\mathbf{A}_9)\,\widehat{\Pi}_3(\mathbf{A}_9)\left(\tau_{3,P2}-\tau_{3,P1}\delta_3^{P1,P2}\right)$$

$$r_{3,P2}=r_{3,P1}\delta_3^{P1,P2}+EV(\mathbf{A}_9)\left(\tau_{3,P2}-\tau_{3,P1}\delta_3^{P1,P2}\right)$$

$$\alpha_3^{P1,P2}=\frac{r_{3,P2}}{r_{3,P1}}=\delta_3^{P1,P2}+EV(\mathbf{A}_9)\left(\frac{\tau_{3,P2}-\tau_{3,P1}\delta_3^{P1,P2}}{r_{3,P1}}\right)$$

Because product $P2$ earns a higher profit than $P1$ in each state, $r_{3,P2}>r_{3,P1}$, and therefore the firm has $\frac{r_{3,P2}}{r_{3,P1}}>1$. If $\alpha_3^{P1,P2}\leq 1$, it implies that $EV(\mathbf{A}_{11}=[P2,P1,P2,M1])\geq EV(\mathbf{A}_9=[P2,P1,P1,M1])$ and $a_3^*=P2$. However, if $\alpha_3^{P1,P2}>1$, then the firm needs to compare the ratio of profits earned from producing $P1$ and $P2$ in state 3, $\frac{r_{3,P2}}{r_{3,P1}}$, with the critical ratio $\alpha_3^{P1,P2}$. a) When $\frac{r_{3,P2}}{r_{3,P1}}>\alpha_3^{P1,P2}$, then $EV(\mathbf{A}_{11}=[P2,P1,P2,M1])>EV(\mathbf{A}_9=[P2,P1,P1,M1])$ and $a_3^*=P2$. b) If $\frac{r_{3,P2}}{r_{3,P1}}<\alpha_3^{P1,P2}$, then $EV(\mathbf{A}_{11}=[P2,P1,P2,M1])<EV(\mathbf{A}_9=[P2,P1,P1,M1])$ and $a_3^*=P1$. c) When $\frac{r_{3,P2}}{r_{3,P1}}=\alpha_3^{P1,P2}$, then $EV(\mathbf{A}_{11}=[P2,P1,P2,M1])=EV(\mathbf{A}_9=[P2,P1,P1,M1])$ and the firm is indifferent between producing $P1$ or $P2$ in state 3.

Proof of Proposition 3: We first derive the critical ratio expression by equating the expected values of the two policies $\mathbf{A}_{25}=[P2,P1,M1,M1]$ and $\mathbf{A}_{27}=[P2,P1,M2,M1]$. Recall that

$$\widehat{\Pi}_1(\mathbf{A}_{27}) = \widehat{\Pi}_1(\mathbf{A}_{25})\times\delta_3^{M1,M2}, \quad (24)$$
$$\widehat{\Pi}_2(\mathbf{A}_{27}) = \widehat{\Pi}_2(\mathbf{A}_{25})\times\delta_3^{M1,M2}, \quad (25)$$
$$\widehat{\Pi}_3(\mathbf{A}_{27}) = \widehat{\Pi}_3(\mathbf{A}_{25}), \quad (26)$$
$$\widehat{\Pi}_4(\mathbf{A}_{27}) = \widehat{\Pi}_4(\mathbf{A}_{25})\times\delta_3^{M1,M2}. \quad (27)$$

Using these relationships, the policies can be expressed as follows:

$$\begin{aligned}
EV(\mathbf{A}_{27}=[P2,P1,M2,M1]) &= EV(\mathbf{A}_{25}=[P2,P1,M1,M1])\\
\frac{\begin{bmatrix} r_{1,P2}\Pi_1(\mathbf{A}_{27})+r_{2,P1}\Pi_2(\mathbf{A}_{27})\\ -c_{3,M2}\Pi_3(\mathbf{A}_{27})-c_{4,M1}\Pi_4(\mathbf{A}_{27})\end{bmatrix}}{\begin{bmatrix} \tau_{1,P2}\Pi_1(\mathbf{A}_{27})+\tau_{2,P1}\Pi_2(\mathbf{A}_{27})\\ +\tau_{3,M2}\Pi_3(\mathbf{A}_{27})+\tau_{4,M1}\Pi_4(\mathbf{A}_{27})\end{bmatrix}} &= \frac{\begin{bmatrix} r_{1,P2}\Pi_1(\mathbf{A}_{25})+r_{2,P1}\Pi_2(\mathbf{A}_{25})\\ -c_{3,M1}\Pi_3(\mathbf{A}_{25})-c_{4,M1}\Pi_4(\mathbf{A}_{25})\end{bmatrix}}{\begin{bmatrix} \tau_{1,P2}\Pi_1(\mathbf{A}_{25})+\tau_{2,P1}\Pi_2(\mathbf{A}_{25})\\ +\tau_{3,M1}\Pi_3(\mathbf{A}_{25})+\tau_{4,M1}\Pi_4(\mathbf{A}_{25})\end{bmatrix}}\\
\frac{\begin{bmatrix} r_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{27})+r_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{27})\\ -c_{3,M2}\widehat{\Pi}_3(\mathbf{A}_{27})-c_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{27})\end{bmatrix}}{\begin{bmatrix} \tau_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{27})+\tau_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{27})\\ +\tau_{3,M2}\widehat{\Pi}_3(\mathbf{A}_{27})+\tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{27})\end{bmatrix}} &= \frac{\begin{bmatrix} r_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25})+r_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25})\\ -c_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25})-c_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25})\end{bmatrix}}{\begin{bmatrix} \tau_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25})+\tau_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25})\\ +\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25})+\tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25})\end{bmatrix}}
\end{aligned}$$

Substituting Using (24)–(27) into $EV(\mathbf{A}_{27}=[P2,P1,M2,M1])$ provides the following:

$$\begin{aligned}
&\frac{r_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25})\delta_3^{M1,M2}+r_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25})\delta_3^{M1,M2}-c_{3,M2}\widehat{\Pi}_3(\mathbf{A}_{25})-c_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25})\delta_3^{M1,M2}}{\tau_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25})\delta_3^{M1,M2}+\tau_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25})\delta_3^{M1,M2}+\tau_{3,M2}\widehat{\Pi}_3(\mathbf{A}_{25})+\tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25})\delta_3^{M1,M2}}\\
=\;&\frac{r_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25})+r_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25})-c_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25})-c_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25})}{\tau_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25})+\tau_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25})+\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25})+\tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25})}
\end{aligned}$$

Adding and subtracting $\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25})\delta_3^{M1,M2}$ into the denominator of $EV(\mathbf{A}_{27}=[P2,P1,M2,M1])$ provides:

$$\begin{aligned}
&r_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25})\delta_3^{M1,M2}+r_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25})\delta_3^{M1,M2}-c_{3,M2}\widehat{\Pi}_3(\mathbf{A}_{25})-c_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25})\delta_3^{M1,M2}\\
=\;&r_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25})\delta_3^{M1,M2}+r_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25})\delta_3^{M1,M2}-c_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25})\delta_3^{M1,M2}-c_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25})\delta_3^{M1,M2}\\
&+EV(\mathbf{A}_{25})\widehat{\Pi}_3(\mathbf{A}_{25})\left(\tau_{3,M2}-\tau_{3,M1}\delta_3^{M1,M2}\right)
\end{aligned}$$

$$-c_{3,M2}\widehat{\Pi}_3(\mathbf{A}_{25})=-c_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25})\delta_3^{M1,M2}+EV(\mathbf{A}_{25})\widehat{\Pi}_3(\mathbf{A}_{25})\left(\tau_{3,M2}-\tau_{3,M1}\delta_3^{M1,M2}\right)$$

$$-c_{3,M2}=-c_{3,M1}\delta_3^{M1,M2}+EV(\mathbf{A}_{25})\left(\tau_{3,M2}-\tau_{3,M1}\delta_3^{M1,M2}\right)$$

$$c_{3,M2}=c_{3,M1}\delta_3^{M1,M2}+EV(\mathbf{A}_{25})\left(\tau_{3,M1}\delta_3^{M1,M2}-\tau_{3,M2}\right)$$

$$\lambda_3^{M1,M2}=\frac{c_{3,M2}}{c_{3,M1}}=\delta_3^{M1,M2}+EV(\mathbf{A}_{25})\left(\frac{\tau_{3,M1}\delta_3^{M1,M2}-\tau_{3,M2}}{c_{3,M1}}\right)$$

Because the cost of the maintenance action $M2$ is higher than that of $M1$, i.e., $c_{3,M2}>c_{3,M1}$, the firm has $\frac{c_{3,M2}}{c_{3,M1}}>1$. If $\lambda_3^{M1,M2}\leq 1$, it implies that $EV(\mathbf{A}_{25}=[P2,P1,M1,M1])\geq EV(\mathbf{A}_{27}=[P2,P1,M2,M1])$ and $a_3^*=M1$. However, if $\lambda_3^{M1,M2}>1$, then the firm needs to compare the ratio of the maintenance costs stemming from $M1$ and $M2$, $\frac{c_{2,M2}}{c_{2,M1}}$, with the critical ratio $\lambda_3^{M1,M2}$. a) When $\frac{c_{3,M2}}{c_{3,M1}}>\lambda_3^{M1,M2}$, then $EV(\mathbf{A}_{25}=[P2,P1,M1,M1])>EV(\mathbf{A}_{27}=[P2,P1,M2,M1])$ and $a_3^*=M1$. b) If $\frac{c_{3,M2}}{c_{3,M1}}<\lambda_3^{M1,M2}$, then $EV(\mathbf{A}_{25}=[P2,P1,M1,M1])<EV(\mathbf{A}_{27}=[P2,P1,M2,M1])$ and $a_3^*=M2$. c) When $\frac{c_{3,M2}}{c_{3,M1}}=\lambda_3^{M1,M2}$, then $EV(\mathbf{A}_{25}=[P2,P1,M1,M1])=EV(\mathbf{A}_{27}=[P2,P1,M2,M1])$ and the firm is indifferent between the maintenance actions $M1$ and $M2$ in state 3.

Proof of Proposition 4: Before we proceed with the proof, let us first establish how the combination of the critical ratios can determine the decision-maker's preferences.

1. If $\frac{r_{3,P2}}{c_{3,M1}}>\gamma_3^{M1,P1}\times\alpha_3^{P1,P2}$, then the policy $\mathbf{A}_{11}=[P2,P1,P2,M1]$ is preferred over the policy $\mathbf{A}_{25}=[P2,P1,M1,M1]$; otherwise, if $\frac{r_{3,P2}}{c_{3,M1}}<\gamma_3^{M1,P1}\times\alpha_3^{P1,P2}$, then the policy $\mathbf{A}_{25}=[P2,P1,M1,M1]$ is preferred. The proof of this statement follows from Propositions 1 and 2.
2. If $\frac{r_{3,P1}}{c_{3,M2}}>\frac{\gamma_3^{M1,P1}}{\lambda_3^{M1,M2}}$, then the policy $\mathbf{A}_9=[P2,P1,P1,M1]$ is preferred over the policy $\mathbf{A}_{27}=[P2,P1,M2,M1]$; otherwise, if $\frac{r_{3,P1}}{c_{3,M2}}<\frac{\gamma_3^{M1,P1}}{\lambda_3^{M1,M2}}$, then the policy $\mathbf{A}_{27}=[P2,P1,M2,M1]$ is preferred. The proof of this statement follows from Propositions 1 and 3.
3. If $\frac{r_{3,P2}}{c_{3,M2}}>\frac{\gamma_3^{M1,P1}}{\lambda_3^{M1,M2}}\times\alpha_3^{P1,P2}$, then the policy $\mathbf{A}_{11}=[P2,P1,P2,M1]$ is preferred over the policy $\mathbf{A}_{27}=[P2,P1,M2,M1]$; otherwise, if $\frac{r_{3,P2}}{c_{3,M2}}<\frac{\gamma_3^{M1,P1}}{\lambda_3^{M1,M2}}\times\alpha_3^{P1,P2}$, then the policy$\mathbf{A}_{27}=[P2,P1,M2,M1]$ is preferred. This statement follows from Propositions 1, 2 and 3.

The above three statements provide the preferences among these four policies. Using the combination of critical ratios, the firm can then determine the optimal policy by comparing the profit earned by manufacturing actions with the associated maintenance costs.

a) From Proposition 1, when $\frac{r_{3,P2}}{r_{3,P1}} < \alpha_3^{P1,P2}$, or when $r_{3,P1} > \frac{r_{3,P2}}{\alpha_3^{P1,P2}}$, $EV(\mathbf{A}_9 = [P2, P1, P1, M1]) > EV(\mathbf{A}_{11} = [P2, P1, P2, M1])$. From Proposition 1, when $r_{3,P1} > c_{3,M1}\gamma_3^{M1,P1}$, $EV(\mathbf{A}_9 = [P2, P1, P1, M1]) > EV(\mathbf{A}_{25} = [P2, P1, M1, M1])$. From Proposition 3, when $\frac{c_{3,M2}}{c_{3,M1}} = \lambda_3^{M1,M2}$, we have $EV(\mathbf{A}_{25} = [P2, P1, M1, M1])$ $= EV(\mathbf{A}_{27} = [P2, P1, M2, M1])$; therefore, when $r_{3,P1} > c_{3,M2}\frac{\gamma_3^{M1,P1}}{\lambda_3^{M1,M2}}$, $EV(\mathbf{A}_9 = [P2, P1, P1, M1]) > EV(\mathbf{A}_{27} = [P2, P1, M2, M1])$. Thus, $\mathbf{A}_9 = [P2, P1, P1, M1]$ is the best policy when $r_{3,P1} \geq \left\{r_{3,P2}\frac{1}{\alpha_3^{P1,P2}}, c_{3,M1}\gamma_3^{M1,P1}, c_{3,M2}\frac{\gamma_3^{M1,P1}}{\lambda_3^{M1,M2}}\right\}$.

b) From Proposition 2, when $\frac{r_{3,P2}}{r_{3,P1}} > \alpha_3^{P1,P2}$, or when $r_{3,P2} > r_{3,P1}\alpha_3^{P1,P2}$, $EV(\mathbf{A}_{11} = [P2, P1, P2, M1]) > EV(\mathbf{A}_9 = [P2, P1, P1, M1])$. From Proposition 1, when $r_{3,P1} = c_{3,M1}\gamma_3^{M1,P1}$, $EV(\mathbf{A}_9 = [P2, P1, P1, M1]) = EV(\mathbf{A}_{25} = [P2, P1, M1, M1])$. Therefore, when $r_{3,P2} > c_{3,M1}\gamma_3^{M1,P1}\alpha_3^{P1,P2}$, $EV(\mathbf{A}_{11} = [P2, P1, P2, M1]) > EV(\mathbf{A}_{25} = [P2, P1, M1, M1])$. From Proposition 3, when $\frac{c_{3,M2}}{c_{3,M1}} = \lambda_3^{M1,M2}$, we have $EV(\mathbf{A}_{25} = [P2, P1, M1, M1])$ $= EV(\mathbf{A}_{27} = [P2, P1, M2, M1])$; therefore, when $r_{3,P2} > c_{3,M2}\frac{\gamma_3^{M1,P1}}{\lambda_3^{M1,M2}}\alpha_2^{P1,P2}$, $EV(\mathbf{A}_{11} = [P2, P1, P2, M1]) > EV(\mathbf{A}_{27} = [P2, P1, M2, M1])$. Thus, $\mathbf{A}_{11} = [P2, P1, P2, M1]$ is the best policy when $r_{3,P2} \geq \left\{r_{3,P1}\alpha_3^{P1,P2}, c_{3,M1}\gamma_3^{M1,P1}\alpha_3^{P1,P2}, c_{3,M2}\frac{\gamma_3^{M1,P1}}{\lambda_3^{M1,M2}}\alpha_3^{P1,P2}\right\}$.

c) From Proposition 1, when $c_{3,M1} > \frac{r_{3,P1}}{\gamma_3^{M1,P1}}$, $EV(\mathbf{A}_{25} = [P2, P1, M1, M1]) > EV(\mathbf{A}_9 = [P2, P1, P1, M1])$. From Proposition 2, when $\frac{r_{3,P2}}{r_{3,P1}} = \alpha_3^{P1,P2}$, $EV(\mathbf{A}_9 = [P2, P1, P1, M1]) = EV(\mathbf{A}_{11} = [P2, P1, P2, M1])$; therefore, when $c_{3,M1} > \frac{r_{3,P2}}{\gamma_3^{M1,P1}\alpha_3^{P1,P2}}$, $EV(\mathbf{A}_{25} = [P2, P1, M1, M1]) > EV(\mathbf{A}_{11} = [P2, P1, P2, M1])$. From Proposition 3, when $\frac{c_{3,M2}}{c_{3,M1}} > \lambda_3^{M1,M2}$, we have $EV(\mathbf{A}_{25} = [P2, P1, M1, M1]) > EV(\mathbf{A}_{27} = [P2, P1, M2, M1])$. Thus, the best policy is $\mathbf{A}_{25} = [P2, P1, M1, M1]$ when $c_{3,M1} \geq \left\{r_{3,P1}\frac{1}{\gamma_3^{M1,P1}}, r_{3,P2}\frac{1}{\gamma_3^{M1,P1}\alpha_3^{P1,P2}}\right\}$ and $c_{3,M1} \leq c_{3,M2}\frac{1}{\lambda_3^{M1,M2}}$.

d) From Proposition 3, when $c_{3,M2} < c_{3,M1}\lambda_3^{M1,M2}$, we have $EV(\mathbf{A}_{27} = [P2, P1, M2, M1]) > EV(\mathbf{A}_{25} = [P2, P1, M1, M1])$. From Proposition 1, when $c_{3,M1} = \frac{r_{3,P1}}{\gamma_3^{M1,P1}}$, $EV(\mathbf{A}_{25} = [P2, P1, M1, M1]) = EV(\mathbf{A}_9 = [P2, P1, P1, M1])$; therefore, when $c_{3,M2} > r_{3,P1}\frac{\lambda_3^{M1,M2}}{\gamma_3^{M1,P1}}$, $EV(\mathbf{A}_{27} = [P2, P1, M2, M1]) > EV(\mathbf{A}_9 = [P2, P1, P1, M1])$. From Proposition 2, when $\frac{r_{3,P2}}{r_{3,P1}} = \alpha_3^{P1,P2}$, $EV(\mathbf{A}_9 = [P2, P1, P1, M1]) = EV(\mathbf{A}_{11} = [P2, P1, P2, M1])$; therefore, when $c_{3,M2} > r_{3,P2}\frac{\lambda_3^{M1,M2}}{\gamma_3^{M1,P1}\alpha_3^{P1,P2}}$, $EV(\mathbf{A}_{27} = [P2, P1, M2, M1]) > EV(\mathbf{A}_{11} = [P2, P1, P2, M1])$. Thus, the best policy is $\mathbf{A}_{27} = [P2, P1, M2, M1]$ when $c_{3,M2} \geq \left\{r_{3,P1}\frac{\lambda_3^{M1,M2}}{\gamma_3^{M1,P1}}, r_{3,P2}\frac{\lambda_3^{M1,M2}}{\gamma_3^{M1,P1}\alpha_3^{P1,P2}}\right\}$ and $c_{3,M2} \leq c_{3,M1}\lambda_3^{M1,M2}$.

Proof of Proposition 5: a) The condition that generates a higher expected production from policy $\mathbf{A}_7 = [P1, P2, P2, M1]$ than $\mathbf{A}_{21} = [P1, P2, M1, M1]$ provides the result.

$$Y_{P1}(\mathbf{A}_7) > Y_{P1}(\mathbf{A}_{21})$$

$$\frac{y_{2,P2}\Pi_2(\mathbf{A}_7) + y_{3,P2}\Pi_3(\mathbf{A}_7)}{\begin{bmatrix} \tau_{1,P1}\Pi_1(\mathbf{A}_7) + \tau_{2,P2}\Pi_2(\mathbf{A}_7) \\ +\tau_{3,P2}\Pi_3(\mathbf{A}_7) + \tau_{4,M1}\Pi_4(\mathbf{A}_7) \end{bmatrix}} > \frac{y_{2,P2}\Pi_2(\mathbf{A}_{21})}{\begin{bmatrix} \tau_{1,P1}\Pi_1(\mathbf{A}_{21}) + \tau_{2,P2}\Pi_2(\mathbf{A}_{21}) \\ +\tau_{3,M1}\Pi_3(\mathbf{A}_{21}) + \tau_{4,M1}\Pi_4(\mathbf{A}_{21}) \end{bmatrix}}$$

$$\frac{y_{2,P2}\widehat{\Pi}_2(\mathbf{A}_7) + y_{3,P2}\widehat{\Pi}_3(\mathbf{A}_7)}{\begin{bmatrix} \tau_{1,P1}\widehat{\Pi}_1(\mathbf{A}_7) + \tau_{2,P2}\widehat{\Pi}_2(\mathbf{A}_7) \\ +\tau_{3,P2}\widehat{\Pi}_3(\mathbf{A}_7) + \tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_7) \end{bmatrix}} > \frac{y_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21})}{\begin{bmatrix} \tau_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21}) + \tau_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21}) \\ +\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{21}) + \tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{21}) \end{bmatrix}}$$

Substituting (17), (18) and (19) into $Y_{P2}(\mathbf{A}_7)$ provides:

$$\frac{y_{2,P2}\left(\widehat{\Pi}_2(\mathbf{A}_{21})-\Delta_{2,3}^{M1,P2}\right)+y_{3,P2}\widehat{\Pi}_3(\mathbf{A}_{21})}{\tau_{1,P1}\left(\widehat{\Pi}_1(\mathbf{A}_{21})-\Delta_{1,3}^{M1,P1}\right)+\tau_{2,P2}\left(\widehat{\Pi}_2(\mathbf{A}_{21})-\Delta_{2,3}^{M1,P2}\right)+\tau_{3,P2}\widehat{\Pi}_3(\mathbf{A}_{21})+\tau_{4,M1}\left(\widehat{\Pi}_4(\mathbf{A}_{21})+\Delta_{4,3}^{M1,P2}\right)}>$$

$$\frac{y_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21})}{\tau_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21})+\tau_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21})+\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{21})+\tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{21})}$$

By adding and subtracting the terms $\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{21})$ in the denominator of $Y_{P2}(\mathbf{A}_7)$ provides:

$$\begin{aligned}&y_{2,P2}\left(\widehat{\Pi}_2(\mathbf{A}_{21})-\Delta_{2,3}^{M1,P2}\right)+y_{3,P2}\widehat{\Pi}_3(\mathbf{A}_{21})\\ >\;& y_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21})+Y_{P2}(\mathbf{A}_{21})\left\{-\tau_{1,P1}\Delta_{1,3}^{M1,P2}-\tau_{2,P2}\Delta_{2,3}^{M1,P2}+(\tau_{3,P2}-\tau_{3,M1})\widehat{\Pi}_3(\mathbf{A}_{21})+\tau_{4,M1}\Delta_{4,3}^{M1,P2}\right\}\end{aligned}$$

$$y_{3,P2}\widehat{\Pi}_3(\mathbf{A}_{21})-y_{2,P2}\Delta_{2,3}^{M1,P2}>Y_{P2}(\mathbf{A}_{21})\left\{\begin{array}{c}-\tau_{1,P1}\Delta_{1,3}^{M1,P2}-\tau_{2,P2}\Delta_{2,3}^{M1,P2}\\+(\tau_{3,P2}-\tau_{3,M1})\widehat{\Pi}_3(\mathbf{A}_{21})+\tau_{4,M1}\Delta_{4,3}^{M1,P2}\end{array}\right\}$$

$$y_{3,P2}-y_{2,P2}\left(\frac{\Delta_{2,3}^{M1,P2}}{\widehat{\Pi}_3(\mathbf{A}_{21})}\right)>Y_{P2}(\mathbf{A}_{21})\left\{\begin{array}{c}-\tau_{1,P1}\left(\frac{\Delta_{1,3}^{M1,P2}}{\widehat{\Pi}_3(\mathbf{A}_{21})}\right)-\tau_{2,P2}\left(\frac{\Delta_{2,3}^{M1,P2}}{\widehat{\Pi}_3(\mathbf{A}_{21})}\right)\\+(\tau_{3,P2}-\tau_{3,M1})+\tau_{4,M1}\left(\frac{\Delta_{4,3}^{M1,P2}}{\widehat{\Pi}_3(\mathbf{A}_{21})}\right)\end{array}\right\}.$$

b) The condition that generates a higher expected production from policy $\mathbf{A}_{23}=[P1,P2,M2,M1]$ than $\mathbf{A}_{21}=[P1,P2,M1,M1]$ provides the result.

$$\begin{aligned}Y_{P2}(\mathbf{A}_{23})&>Y_{P2}(\mathbf{A}_{21})\\ \frac{y_{2,P2}\Pi_2(\mathbf{A}_{23})}{\left[\begin{array}{c}\tau_{1,P1}\Pi_1(\mathbf{A}_{23})+\tau_{2,P2}\Pi_2(\mathbf{A}_{23})\\+\tau_{3,M2}\Pi_3(\mathbf{A}_{23})+\tau_{4,M1}\Pi_4(\mathbf{A}_{23})\end{array}\right]}&>\frac{y_{2,P2}\Pi_2(\mathbf{A}_{21})}{\left[\begin{array}{c}\tau_{1,P1}\Pi_1(\mathbf{A}_{21})+\tau_{2,P2}\Pi_2(\mathbf{A}_{21})\\+\tau_{3,M1}\Pi_3(\mathbf{A}_{21})+\tau_{4,M1}\Pi_4(\mathbf{A}_{21})\end{array}\right]}\\ \frac{y_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{23})}{\left[\begin{array}{c}\tau_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{23})+\tau_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{23})\\+\tau_{3,M2}\widehat{\Pi}_3(\mathbf{A}_{23})+\tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{23})\end{array}\right]}&>\frac{y_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21})}{\left[\begin{array}{c}\tau_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21})+\tau_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21})\\+\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{21})+\tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{21})\end{array}\right]}\end{aligned}$$

Substituting Using (24)–(27) into Y_{P2} $(\mathbf{A}_{23})$ provides the following:

$$\begin{aligned}&\frac{y_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21})\,\delta_3^{M1,M2}}{\tau_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21})\,\delta_3^{M1,M2}+\tau_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21})\,\delta_3^{M1,M2}+\tau_{3,M2}\widehat{\Pi}_3(\mathbf{A}_{21})+\tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{21})\,\delta_3^{M1,M2}}\\ >\;&\frac{y_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21})}{\tau_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21})+\tau_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21})+\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{21})+\tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{21})}\end{aligned}$$

Adding and subtracting $\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{21})\,\delta_3^{M1,M2}$ into the denominator of $Y_{P2}(\mathbf{A}_{23})$ provides:

$$y_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21})\,\delta_3^{M1,M2}>y_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21})\,\delta_3^{M1,M2}+Y_{P2}(\mathbf{A}_{21})\,\widehat{\Pi}_3(\mathbf{A}_{21})\left(\tau_{3,M2}-\tau_{3,M1}\delta_3^{M1,M2}\right)$$

$$Y_{P2}(\mathbf{A}_{21})\,\widehat{\Pi}_3(\mathbf{A}_{21})\left(\tau_{3,M2}-\tau_{3,M1}\delta_3^{M1,M2}\right)<0.$$

Because $Y_{P2}(\mathbf{A}_{21})$ and $\widehat{\Pi}_3(\mathbf{A}_{21})$ are both positive, the above is satisfied when

$$\left(\tau_{3,M2}-\tau_{3,M1}\delta_3^{M1,M2}\right)<0.$$

A similar condition can be obtained for $P1$.

Proof of Proposition 6: The condition that leads to a higher expected production from policy $\mathbf{A}_{41} = [P2, M1, P1, M1]$ than $\mathbf{A}_{25} = [P2, P1, M1, M1]$ provides the result.

$$\begin{aligned}
Y_{P2}(\mathbf{A}_{41}) &> Y_{P2}(\mathbf{A}_{25}) \\
\frac{y_{1,P2}\Pi_1(\mathbf{A}_{41})}{\left[\begin{array}{c}\tau_{1,P2}\Pi_1(\mathbf{A}_{41}) + \tau_{2,M1}\Pi_2(\mathbf{A}_{41}) \\ +\tau_{3,P1}\Pi_3(\mathbf{A}_{41}) + \tau_{4,M1}\Pi_4(\mathbf{A}_{41})\end{array}\right]} &> \frac{y_{1,P2}\Pi_1(\mathbf{A}_{25})}{\left[\begin{array}{c}\tau_{1,P2}\Pi_1(\mathbf{A}_{25}) + \tau_{2,P1}\Pi_2(\mathbf{A}_{25}) \\ +\tau_{3,M1}\Pi_3(\mathbf{A}_{25}) + \tau_{4,M1}\Pi_4(\mathbf{A}_{25})\end{array}\right]} \\
\frac{y_{1,P2}\widehat{\Pi}_2(\mathbf{A}_{41})}{\left[\begin{array}{c}\tau_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{41}) + \tau_{2,M1}\widehat{\Pi}_2(\mathbf{A}_{41}) \\ +\tau_{3,P1}\widehat{\Pi}_3(\mathbf{A}_{41}) + \tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{41})\end{array}\right]} &> \frac{y_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25})}{\left[\begin{array}{c}\tau_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25}) + \tau_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25}) \\ +\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25}) + \tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25})\end{array}\right]}
\end{aligned}$$

Replicating (17)–(19) for state 2 and then substituting the relationships from states 2 and 3 into $Y_{P2}(\mathbf{A}_{41})$ provide the following:

$$\begin{aligned}
&\frac{y_{1,P2}\left(\widehat{\Pi}_1(\mathbf{A}_{25}) + \Delta_{1,2}^{M1,P1} - \Delta_{1,3}^{M1,P1}\right)}{\left[\begin{array}{c}\tau_{1,P2}\left(\widehat{\Pi}_1(\mathbf{A}_{25}) + \Delta_{1,2}^{M1,P1} - \Delta_{1,3}^{M1,P1}\right) + \tau_{2,M1}\left(\widehat{\Pi}_2(\mathbf{A}_{25}) - \Delta_{2,3}^{M1,P1}\right) \\ +\tau_{3,P1}\left(\widehat{\Pi}_3(\mathbf{A}_{25}) + \Delta_{3,2}^{M1,P1}\right) + \tau_{4,M1}\left(\widehat{\Pi}_4(\mathbf{A}_{25}) + \Delta_{4,3}^{M1,P1} - \Delta_{4,2}^{M1,P1}\right)\end{array}\right]} \\
> \quad &\frac{y_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25})}{\tau_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25}) + \tau_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25}) + \tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25}) + \tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{25})}
\end{aligned}$$

Adding and subtracting $\tau_{2,P1}\widehat{\Pi}_2(\mathbf{A}_{25}) + \tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{25})$ into the denominator of $Y_{P2}(\mathbf{A}_{41})$ provides:

$$\begin{aligned}
y_{1,P2}\left(\widehat{\Pi}_1(\mathbf{A}_{25}) + \Delta_{1,2}^{M1,P1} - \Delta_{1,3}^{M1,P1}\right) &> y_{1,P2}\widehat{\Pi}_1(\mathbf{A}_{25}) \\
&+ Y_{P2}(\mathbf{A}_{25})\left\{\begin{array}{c}\tau_{1,P2}\left(\Delta_{1,2}^{M1,P1} - \Delta_{1,3}^{M1,P1}\right) \\ +(\tau_{2,M1} - \tau_{2,P1})\widehat{\Pi}_2(\mathbf{A}_{25}) - \tau_{2,M1}\Delta_{2,3}^{M1,P1} \\ +(\tau_{3,P1} - \tau_{3,M1})\widehat{\Pi}_3(\mathbf{A}_{25}) + \tau_{3,P1}\Delta_{3,2}^{M1,P1} \\ +\tau_{4,M1}\left(\Delta_{4,3}^{M1,P1} - \Delta_{4,2}^{M1,P1}\right)\end{array}\right\}
\end{aligned}$$

$$y_{1,P2} > \frac{Y_{P2}(\mathbf{A}_{25})}{\left(\Delta_{1,2}^{M1,P1} - \Delta_{1,3}^{M1,P1}\right)}\left\{\begin{array}{c}\tau_{1,P1}\left(\Delta_{1,2}^{M1,P1} - \Delta_{1,3}^{M1,P1}\right) \\ +(\tau_{2,M1} - \tau_{2,P1})\widehat{\Pi}_2(\mathbf{A}_{25}) - \tau_{2,M1}\Delta_{2,3}^{M1,P1} \\ +(\tau_{3,P1} - \tau_{3,M1})\widehat{\Pi}_3(\mathbf{A}_{25}) + \tau_{3,P1}\Delta_{3,2}^{M1,P1} \\ +\tau_{4,M1}\left(\Delta_{4,3}^{M1,P1} - \Delta_{4,2}^{M1,P1}\right)\end{array}\right\}.$$

A similar condition can be obtained for $P1$.

Proof of Proposition 7: The proof follows from the comparison of the numerator terms in steady-state probabilities in (2)–(5). a) In Group 3 policies, product $P1$ is manufactured only in state 3 in policies $\mathbf{A}_{41}$, $\mathbf{A}_{42}$, $\mathbf{A}_{45}$, and $\mathbf{A}_{46}$. In these four policies, the expected production of $P1$ is equal to:

$$Y_{P1}(\mathbf{A}) = y_{3,P1}\Pi_3(\mathbf{A}) / \sum_{i=1}^{N} \tau_{i,a_i}\Pi_i(\mathbf{A}) = y_{3,P1}\widehat{\Pi}_3(\mathbf{A}) / \sum_{i=1}^{N} \tau_{i,a_i}\widehat{\Pi}_i(\mathbf{A}).$$

From (4), it can be observed that $\widehat{\Pi}_3(\mathbf{A}_{42}) \geq \left\{\widehat{\Pi}_3(\mathbf{A}_{41}), \widehat{\Pi}_3(\mathbf{A}_{45}), \widehat{\Pi}_3(\mathbf{A}_{46})\right\}$. Thus, $Y_{P1}(\mathbf{A}_{42}) \geq \{Y_{P1}(\mathbf{A}_{41}),$ $Y_{P1}(\mathbf{A}_{45}), Y_{P1}(\mathbf{A}_{46})\}$, resulting in the highest expected production amount among these four policies. Then, if $MPR_{P1} > Y_{P1}(\mathbf{A}_{42})$, these four policies cannot be optimal. Similarly, product $P2$ is manufactured only in state 3 in policies $\mathbf{A}_{35}$, $\mathbf{A}_{36}$, $\mathbf{A}_{39}$, and $\mathbf{A}_{40}$. In these four policies, the expected production of $P2$ is equal to:

$$Y_{P2}(\mathbf{A}) = y_{3,P2}\Pi_3(\mathbf{A}) / \sum_{i=1}^{N} \tau_{i,a_i}\Pi_i(\mathbf{A}) = y_{3,P2}\widehat{\Pi}_3(\mathbf{A}) / \sum_{i=1}^{N} \tau_{i,a_i}\widehat{\Pi}_i(\mathbf{A}).$$

From (4), it can be observed that $\widehat{\Pi}_3(\mathbf{A}_{36}) \geq \left\{\widehat{\Pi}_3(\mathbf{A}_{35}), \widehat{\Pi}_3(\mathbf{A}_{39}), \widehat{\Pi}_3(\mathbf{A}_{40})\right\}$. Thus, $Y_{P2}(\mathbf{A}_{36}) \geq \{Y_{P2}(\mathbf{A}_{35}), Y_{P2}(\mathbf{A}_{39}), Y_{P2}(\mathbf{A}_{40})\}$, resulting in the highest expected production amount among these four policies. Then, if $MPR_{P2} > Y_{P2}(\mathbf{A}_{36})$, these four policies cannot be optimal. Collectively, the conditions $MPR_{P1} > Y_{P1}(\mathbf{A}_{42})$ and $MPR_{P2} > Y_{P2}(\mathbf{A}_{36})$ eliminate all eight policies in Group 3 from the list of potentially optimal policies, reducing its set to 20 policies. The consequence of eliminating Group 3 policies is that the firm has to commit to production in the first two states. As a result, in state 2, the decision maker does not have to use the two sets of critical ratios $\gamma_2^{M1,P1}$ and $\lambda_2^{M1,M2}$. Like state 1, in state 2, the only comparison should be made using the critical ratio $\alpha_2^{P1,P2}$ in order to determine which product should be manufactured.
b) Among Group 2 policies, product $P1$ is manufactured in state 1 in policies $\mathbf{A}_{21}$ through $\mathbf{A}_{24}$. In these four policies, the expected production of $P1$ is equal to:

$$Y_{P1}(\mathbf{A}) = y_{1,P1}\Pi_1(\mathbf{A}) / \sum_{i=1}^{N} \tau_{i,a_i}\Pi_i(\mathbf{A}) = y_{1,P1}\widehat{\Pi}_1(\mathbf{A}) / \sum_{i=1}^{N} \tau_{i,a_i}\widehat{\Pi}_i(\mathbf{A}).$$

From (2), it can be observed that $\widehat{\Pi}_1(\mathbf{A}_{24}) \geq \left\{\widehat{\Pi}_1(\mathbf{A}_{21}), \widehat{\Pi}_1(\mathbf{A}_{22}), \widehat{\Pi}_1(\mathbf{A}_{23})\right\}$. Thus, $Y_{P1}(\mathbf{A}_{24}) \geq \{Y_{P1}(\mathbf{A}_{21}), Y_{P1}(\mathbf{A}_{22}), Y_{P1}(\mathbf{A}_{23})\}$, resulting in the highest expected production amount among these four policies. Then, if $MPR_{P1} > Y_{P1}(\mathbf{A}_{24})$, these four policies cannot be optimal. Similarly, product $P2$ is manufactured only in state 1 in policies $\mathbf{A}_{25}$ through $\mathbf{A}_{28}$. In these four policies, the expected production of $P2$ is equal to:

$$Y_{P2}(\mathbf{A}) = y_{1,P2}\Pi_1(\mathbf{A}) / \sum_{i=1}^{N} \tau_{i,a_i}\Pi_i(\mathbf{A}) = y_{1,P2}\widehat{\Pi}_1(\mathbf{A}) / \sum_{i=1}^{N} \tau_{i,a_i}\widehat{\Pi}_i(\mathbf{A}).$$

From (2), it can be observed that $\widehat{\Pi}_1(\mathbf{A}_{28}) \geq \left\{\widehat{\Pi}_1(\mathbf{A}_{25}), \widehat{\Pi}_1(\mathbf{A}_{26}), \widehat{\Pi}_1(\mathbf{A}_{27})\right\}$. Thus, $Y_{P2}(\mathbf{A}_{28}) \geq \{Y_{P2}(\mathbf{A}_{25}), Y_{P2}(\mathbf{A}_{26}), Y_{P2}(\mathbf{A}_{27})\}$, resulting in the highest expected production amount among these four policies. Then, if $MPR_{P2} > Y_{P2}(\mathbf{A}_{28})$, these four policies cannot be optimal. Collectively, the conditions $MPR_{P1} > Y_{P1}(\mathbf{A}_{24})$ and $MPR_{P2} > Y_{P2}(\mathbf{A}_{28})$ eliminate all eight policies in Group 2 from the list of potentially optimal policies, reducing its set to 20 policies. The consequence of eliminating Group 2 policies is that the firm has to commit to production in the third state. As a result, in state 3, the decision maker does not have to use the two sets of critical ratios $\gamma_3^{M1,P1}$ and $\lambda_3^{M1,M2}$. Like state 1, in state 3, the only comparison should be made using the critical ratio $\alpha_2^{P1,P2}$ in order to determine which product should be manufactured.
c) In Group 3 policies, product $P1$ is manufactured only in state 1 in policies $\mathbf{A}_{35}$, $\mathbf{A}_{36}$, $\mathbf{A}_{39}$, and $\mathbf{A}_{40}$. In these four policies, the expected production of $P1$ is equal to:

$$Y_{P1}(\mathbf{A}) = y_{1,P1}\Pi_1(\mathbf{A}) / \sum_{i=1}^{N} \tau_{i,a_i}\Pi_i(\mathbf{A}) = y_{1,P1}\widehat{\Pi}_1(\mathbf{A}) / \sum_{i=1}^{N} \tau_{i,a_i}\widehat{\Pi}_i(\mathbf{A}).$$

From (2), it can be observed that $\widehat{\Pi}_1(\mathbf{A}_{40}) \geq \left\{\widehat{\Pi}_1(\mathbf{A}_{35}), \widehat{\Pi}_1(\mathbf{A}_{36}), \widehat{\Pi}_1(\mathbf{A}_{39})\right\}$. Moreover, again from (2), it can be verified that $\widehat{\Pi}_1(\mathbf{A}_{40}) \geq \left\{\widehat{\Pi}_3(\mathbf{A}_{41}), \widehat{\Pi}_3(\mathbf{A}_{42}), \widehat{\Pi}_3(\mathbf{A}_{45}), \widehat{\Pi}_3(\mathbf{A}_{46})\right\}$. It is already known that $y_{1,P1} > y_{3,P1}$; thus, $Y_{P1}(\mathbf{A}_{40}) \geq \{Y_{P1}(\mathbf{A}_{35}), Y_{P1}(\mathbf{A}_{36}), Y_{P1}(\mathbf{A}_{39}), Y_{P1}(\mathbf{A}_{41}), Y_{P1}(\mathbf{A}_{42}), Y_{P1}(\mathbf{A}_{45}), Y_{P1}(\mathbf{A}_{46})\}$. From the comparison of (2) among policies in Groups 2 and 3 and from the analysis in part b, it can be observed

that $\widehat{\Pi}_1(\mathbf{A}_{40}) \geq \widehat{\Pi}_1(\mathbf{A}_{24}) \geq \left\{\widehat{\Pi}_1(\mathbf{A}_{21}), \widehat{\Pi}_1(\mathbf{A}_{22}), \widehat{\Pi}_1(\mathbf{A}_{23})\right\} \geq \left\{\widehat{\Pi}_2(\mathbf{A}_{25}), \widehat{\Pi}_2(\mathbf{A}_{26}), \widehat{\Pi}_2(\mathbf{A}_{27}), \widehat{\Pi}_2(\mathbf{A}_{28})\right\}$; thus, $Y_{P1}(\mathbf{A}_{40}) \geq \{Y_{P1}(\mathbf{A}_{21}), Y_{P1}(\mathbf{A}_{22}), Y_{P1}(\mathbf{A}_{23}), Y_{P1}(\mathbf{A}_{24}), Y_{P1}(\mathbf{A}_{25}), Y_{P1}(\mathbf{A}_{26}), Y_{P1}(\mathbf{A}_{27}), Y_{P1}(\mathbf{A}_{28})\}$. This implies that the expected production of $P1$ is maximized in policy $\mathbf{A}_{40}$ among all policies in Groups 2 and 3. Then, if $MPR_{P1} > Y_{P1}(\mathbf{A}_{40})$, no policy in Groups 2 and 3 can be optimal, reducing the set of potentially optimal policies to those in Group 1 (with 12 policies). The consequence of this result is that manufacturing has to take place in states 1, 2, and 3. As a result, the decision maker can eliminate the critical ratios $\gamma_2^{M1,P1}$, $\lambda_2^{M1,M2}$, $\gamma_3^{M1,P1}$ and $\lambda_3^{M1,M2}$. The decision is now limited to the comparison of the production-related critical ratios $\alpha_1^{P1,P2}$, $\alpha_2^{P1,P2}$, and $\alpha_3^{P1,P2}$ in states 1,2, and 3, and $\lambda_4^{M1,M2}$ in state 4.
Similarly, product $P2$ is manufactured only in state 1 in Group 3 policies $\mathbf{A}_{41}$, $\mathbf{A}_{42}$, $\mathbf{A}_{45}$, and $\mathbf{A}_{46}$. In these four policies, the expected production of $P2$ is equal to:

$$Y_{P2}(\mathbf{A}) = y_{1,P2}\Pi_1(\mathbf{A}) / \sum_{i=1}^{N} \tau_{i,a_i}\Pi_i(\mathbf{A}) = y_{1,P2}\widehat{\Pi}_1(\mathbf{A}) / \sum_{i=1}^{N} \tau_{i,a_i}\widehat{\Pi}_i(\mathbf{A}).$$

From (2), it can be observed that $\widehat{\Pi}_1(\mathbf{A}_{46}) \geq \left\{\widehat{\Pi}_1(\mathbf{A}_{41}), \widehat{\Pi}_1(\mathbf{A}_{42}), \widehat{\Pi}_1(\mathbf{A}_{45})\right\}$. Moreover, again from (2), it can be verified that $\widehat{\Pi}_1(\mathbf{A}_{46}) \geq \left\{\widehat{\Pi}_3(\mathbf{A}_{35}), \widehat{\Pi}_3(\mathbf{A}_{36}), \widehat{\Pi}_3(\mathbf{A}_{39}), \widehat{\Pi}_3(\mathbf{A}_{40})\right\}$. It is already known that $y_{1,P2} > y_{3,P2}$; thus, $Y_{P2}(\mathbf{A}_{46}) \geq \{Y_{P2}(\mathbf{A}_{35}), Y_{P2}(\mathbf{A}_{36}), Y_{P2}(\mathbf{A}_{39}), Y_{P2}(\mathbf{A}_{40}), Y_{P2}(\mathbf{A}_{41}), Y_{P2}(\mathbf{A}_{42}), Y_{P2}(\mathbf{A}_{45})\}$. From the comparison of (2) among policies in Groups 2 and 3 and from the analysis in part b, it can be observed that $\widehat{\Pi}_1(\mathbf{A}_{46}) \geq \widehat{\Pi}_1(\mathbf{A}_{28}) \geq \left\{\widehat{\Pi}_1(\mathbf{A}_{25}), \widehat{\Pi}_1(\mathbf{A}_{26}), \widehat{\Pi}_1(\mathbf{A}_{27})\right\} \geq \left\{\widehat{\Pi}_2(\mathbf{A}_{21}), \widehat{\Pi}_2(\mathbf{A}_{22}), \widehat{\Pi}_2(\mathbf{A}_{23}), \widehat{\Pi}_2(\mathbf{A}_{24})\right\}$; thus, $Y_{P2}(\mathbf{A}_{46}) \geq \{Y_{P2}(\mathbf{A}_{21}), Y_{P2}(\mathbf{A}_{22}), Y_{P2}(\mathbf{A}_{23}), Y_{P2}(\mathbf{A}_{24}), Y_{P2}(\mathbf{A}_{25}), Y_{P2}(\mathbf{A}_{26}), Y_{P2}(\mathbf{A}_{27}), Y_{P2}(\mathbf{A}_{28})\}$. This implies that the expected production of $P2$ is maximized in policy $\mathbf{A}_{46}$ among all policies in Groups 2 and 3. Then, if $MPR_{P2} > Y_{P2}(\mathbf{A}_{46})$, no policy in Groups 2 and 3 can be optimal, reducing the set of potentially optimal policies to those in Group 1 (with 12 policies). The consequence of this result is that manufacturing has to take place in states 1, 2, and 3. As a result, the decision maker can eliminate the critical ratios $\gamma_2^{M1,P1}$, $\lambda_2^{M1,M2}$, $\gamma_3^{M1,P1}$ and $\lambda_3^{M1,M2}$. The decision is now limited to the comparison of the production-related critical ratios $\alpha_1^{P1,P2}$, $\alpha_2^{P1,P2}$, and $\alpha_3^{P1,P2}$ in states 1,2, and 3, and $\lambda_4^{M1,M2}$ in state 4.
d) From (2), it can be observed that $\widehat{\Pi}_1(\mathbf{A}_{40}) \geq \{\widehat{\Pi}_1(\mathbf{A}_3), \widehat{\Pi}_1(\mathbf{A}_4), \widehat{\Pi}_1(\mathbf{A}_5), \widehat{\Pi}_1(\mathbf{A}_6), \widehat{\Pi}_1(\mathbf{A}_7), \widehat{\Pi}_1(\mathbf{A}_8)\}$ and therefore, $Y_{P1}(\mathbf{A}_{40}) \geq \{Y_{P1}(\mathbf{A}_3), Y_{P1}(\mathbf{A}_4), Y_{P1}(\mathbf{A}_5), Y_{P1}(\mathbf{A}_6), Y_{P1}(\mathbf{A}_7), Y_{P1}(\mathbf{A}_8)\}$. As a result, when $MPR_{P1} > Y_{P1}(\mathbf{A}_{40})$ policies $\mathbf{A}_3$ through $\mathbf{A}_8$ cannot be optimal. The decision maker can eliminate the critical ratios $\alpha_1^{P1,P2}$, $\gamma_2^{M1,P1}$, $\lambda_2^{M1,M2}$, $\gamma_3^{M1,P1}$ and $\lambda_3^{M1,M2}$ from consideration. The decision is now limited to the comparison of the critical ratios $\alpha_2^{P1,P2}$, and $\alpha_3^{P1,P2}$ in states 2, and 3, and $\lambda_4^{M1,M2}$ in state 4.
Similarly, it can be observed from (2), that $\widehat{\Pi}_1(\mathbf{A}_{46}) \geq \left\{\widehat{\Pi}_1(\mathbf{A}_9), \widehat{\Pi}_1(\mathbf{A}_{10}), \widehat{\Pi}_1(\mathbf{A}_{11}), \widehat{\Pi}_1(\mathbf{A}_{12}), \widehat{\Pi}_1(\mathbf{A}_{13}), \widehat{\Pi}_1(\mathbf{A}_{14})\right\}$ and therefore, $Y_{P2}(\mathbf{A}_{46}) \geq \{Y_{P2}(\mathbf{A}_9), Y_{P2}(\mathbf{A}_{10}), Y_{P2}(\mathbf{A}_{11}), Y_{P2}(\mathbf{A}_{12}), Y_{P2}(\mathbf{A}_{13}), Y_{P2}(\mathbf{A}_{14})\}$. As a result, when $MPR_{P2} > Y_{P2}(\mathbf{A}_{46})$ policies $\mathbf{A}_9$ through $\mathbf{A}_{14}$ cannot be optimal. The decision maker can eliminate the critical ratios $\alpha_1^{P1,P2}$, $\gamma_2^{M1,P1}$, $\lambda_2^{M1,M2}$, $\gamma_3^{M1,P1}$ and $\lambda_3^{M1,M2}$ from consideration. The decision is now limited to the comparison of the critical ratios $\alpha_2^{P1,P2}$, and $\alpha_3^{P1,P2}$ in states 2, and 3, and $\lambda_4^{M1,M2}$ in state 4.

Proof of Proposition 8: a) The condition that generates a higher expected production from policy $\mathbf{A}_{21} = [P1, P2, M1, M1]$ than $\mathbf{A}_5 = [P1, P2, P1, M1]$ provides the result.

$$Y_{P1}(\mathbf{A}_5) < Y_{P1}(\mathbf{A}_{21})$$

$$\frac{y_{1,P1}\Pi_1(\mathbf{A}_5)+y_{3,P1}\Pi_3(\mathbf{A}_5)}{\begin{bmatrix}\tau_{1,P1}\Pi_1(\mathbf{A}_5)+\tau_{2,P2}\Pi_2(\mathbf{A}_5)\\+\tau_{3,P1}\Pi_3(\mathbf{A}_5)+\tau_{4,M1}\Pi_4(\mathbf{A}_5)\end{bmatrix}} < \frac{y_{1,P1}\Pi_1(\mathbf{A}_{21})}{\begin{bmatrix}\tau_{1,P1}\Pi_1(\mathbf{A}_{21})+\tau_{2,P2}\Pi_2(\mathbf{A}_{21})\\+\tau_{3,M1}\Pi_3(\mathbf{A}_{21})+\tau_{4,M1}\Pi_4(\mathbf{A}_{21})\end{bmatrix}}$$

$$\frac{y_{1,P1}\widehat{\Pi}_1(\mathbf{A}_5)+y_{3,P1}\widehat{\Pi}_3(\mathbf{A}_5)}{\begin{bmatrix}\tau_{1,P1}\widehat{\Pi}_1(\mathbf{A}_5)+\tau_{2,P2}\widehat{\Pi}_2(\mathbf{A}_5)\\+\tau_{3,P1}\widehat{\Pi}_3(\mathbf{A}_5)+\tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_5)\end{bmatrix}} < \frac{y_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21})}{\begin{bmatrix}\tau_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21})+\tau_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21})\\+\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{21})+\tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{21})\end{bmatrix}}$$

Substituting (17), (18) and (19) into $Y_{P1}(\mathbf{A}_5)$provides:

$$\frac{y_{1,P1}\left(\widehat{\Pi}_1(\mathbf{A}_{21})-\Delta_{2,3}^{M1,P1}\right)+y_{3,P1}\widehat{\Pi}_3(\mathbf{A}_{21})}{\tau_{1,P1}\left(\widehat{\Pi}_1(\mathbf{A}_{21})-\Delta_{1,3}^{M1,P1}\right)+\tau_{2,P2}\left(\widehat{\Pi}_2(\mathbf{A}_{21})-\Delta_{2,3}^{M1,P1}\right)+\tau_{3,P1}\widehat{\Pi}_3(\mathbf{A}_{21})+\tau_{4,M1}\left(\widehat{\Pi}_4(\mathbf{A}_{21})+\Delta_{4,3}^{M1,P1}\right)} < \frac{y_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21})}{\tau_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21})+\tau_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21})+\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{21})+\tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{21})}$$

By adding and subtracting the terms $\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{21})$ in the denominator of $Y_{P2}(\mathbf{A}_5)$ provides:

$$\begin{aligned}&y_{1,P1}\left(\widehat{\Pi}_1(\mathbf{A}_{21})-\Delta_{2,3}^{M1,P1}\right)+y_{3,P1}\widehat{\Pi}_3(\mathbf{A}_{21})\\ <\ &y_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21})+Y_{P1}(\mathbf{A}_{21})\left\{-\tau_{1,P1}\Delta_{1,3}^{M1,P1}-\tau_{2,P2}\Delta_{2,3}^{M1,P1}+(\tau_{3,P1}-\tau_{3,M1})\widehat{\Pi}_3(\mathbf{A}_{21})+\tau_{4,M1}\Delta_{4,3}^{M1,P1}\right\}\end{aligned}$$

$$y_{3,P1}\widehat{\Pi}_3(\mathbf{A}_{21})-y_{1,P1}\Delta_{2,3}^{M1,P1} < Y_{P1}(\mathbf{A}_{21})\left\{\begin{array}{c}-\tau_{1,P1}\Delta_{1,3}^{M1,P1}-\tau_{2,P2}\Delta_{2,3}^{M1,P1}\\+(\tau_{3,P1}-\tau_{3,M1})\widehat{\Pi}_3(\mathbf{A}_{21})+\tau_{4,M1}\Delta_{4,3}^{M1,P1}\end{array}\right\}$$

$$y_{3,P1}-y_{1,P1}\left(\frac{\Delta_{2,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{21})}\right) < Y_{P1}(\mathbf{A}_{21})\left\{\begin{array}{l}-\tau_{1,P1}\left(\frac{\Delta_{1,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{21})}\right)-\tau_{2,P2}\left(\frac{\Delta_{2,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{21})}\right)\\+(\tau_{3,P1}-\tau_{3,M1})+\tau_{4,M1}\left(\frac{\Delta_{4,3}^{M1,P1}}{\widehat{\Pi}_3(\mathbf{A}_{21})}\right)\end{array}\right\}.$$

b) The condition that generates a lower expected production from policy $\mathbf{A}_{23}=[P1,P2,M2,M1]$ than $\mathbf{A}_{21}=[P1,P2,M1,M1]$ provides the result.

$$Y_{P1}(\mathbf{A}_{23}) < Y_{P1}(\mathbf{A}_{21})$$

$$\frac{y_{1,P1}\Pi_1(\mathbf{A}_{23})}{\begin{bmatrix}\tau_{1,P1}\Pi_1(\mathbf{A}_{23})+\tau_{2,P2}\Pi_2(\mathbf{A}_{23})\\+\tau_{3,M2}\Pi_3(\mathbf{A}_{23})+\tau_{4,M1}\Pi_4(\mathbf{A}_{23})\end{bmatrix}} < \frac{y_{1,P1}\Pi_1(\mathbf{A}_{21})}{\begin{bmatrix}\tau_{1,P1}\Pi_1(\mathbf{A}_{21})+\tau_{2,P2}\Pi_2(\mathbf{A}_{21})\\+\tau_{3,M1}\Pi_3(\mathbf{A}_{21})+\tau_{4,M1}\Pi_4(\mathbf{A}_{21})\end{bmatrix}}$$

$$\frac{y_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{23})}{\begin{bmatrix}\tau_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{23})+\tau_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{23})\\+\tau_{3,M2}\widehat{\Pi}_3(\mathbf{A}_{23})+\tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{23})\end{bmatrix}} < \frac{y_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21})}{\begin{bmatrix}\tau_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21})+\tau_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21})\\+\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{21})+\tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{21})\end{bmatrix}}$$

Substituting (24)–(27) into Y_{P1} $(\mathbf{A}_{23})$ provides the following:

$$\begin{aligned}&\frac{y_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21})\,\delta_3^{M1,M2}}{\tau_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21})\,\delta_3^{M1,M2}+\tau_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21})\,\delta_3^{M1,M2}+\tau_{3,M2}\widehat{\Pi}_3(\mathbf{A}_{21})+\tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{21})\,\delta_3^{M1,M2}}\\ <\ &\frac{y_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21})}{\tau_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21})+\tau_{2,P2}\widehat{\Pi}_2(\mathbf{A}_{21})+\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{21})+\tau_{4,M1}\widehat{\Pi}_4(\mathbf{A}_{21})}\end{aligned}$$

Adding and subtracting $\tau_{3,M1}\widehat{\Pi}_3(\mathbf{A}_{21})\,\delta_3^{M1,M2}$ into the denominator of $Y_{P1}(\mathbf{A}_{23})$ provides:

$$y_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21})\,\delta_3^{M1,M2} < y_{1,P1}\widehat{\Pi}_1(\mathbf{A}_{21})\,\delta_3^{M1,M2}l + Y_{P1}(\mathbf{A}_{21})\,\widehat{\Pi}_3(\mathbf{A}_{21})\left(\tau_{3,M2} - \tau_{3,M1}\delta_3^{M1,M2}\right)$$

$$Y_{P1}(\mathbf{A}_{21})\,\widehat{\Pi}_3(\mathbf{A}_{21})\left(\tau_{3,M2} - \tau_{3,M1}\delta_3^{M1,M2}\right) > 0.$$

Because $Y_{P1}(\mathbf{A}_{21})$ and $\widehat{\Pi}_3(\mathbf{A}_{21})$are both positive, the above is satisfied when

$$\left(\tau_{3,M2} - \tau_{3,M1}\delta_3^{M1,M2}\right) > 0.$$

Proof of Proposition 9: The proof follows from the comparison of the numerator terms in steady-state probabilities in (2)–(5).

a) Among Group 1 policies, notice that $\mathbf{A}_9 = [P2, P1, P1, M1]$ yields smaller expected production of $P1$ than policies $\mathbf{A}_{10}$, $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{13}$, and $\mathbf{A}_{14}$. Similarly, among Group 1 policies, policy $\mathbf{A}_7 = [P1, P2, P2, M1]$ yields smaller expected production of $P2$ than policies $\mathbf{A}_3$, $\mathbf{A}_4$, $\mathbf{A}_5$, $\mathbf{A}_6$, and $\mathbf{A}_8$. When $Y_{P1}(\mathbf{A}_9) > XPR_{P1}$ and $Y_{P2}(\mathbf{A}_7) > XPR_{P2}$, then $\{Y_{P1}(\mathbf{A}_{10}), Y_{P1}(\mathbf{A}_{11}), Y_{P1}(\mathbf{A}_{12}), Y_{P1}(\mathbf{A}_{13}), Y_{P1}(\mathbf{A}_{14})\} > XPR_{P1}$ and$\{Y_{P2}(\mathbf{A}_3), Y_{P2}(\mathbf{A}_4), Y_{P2}(\mathbf{A}_5), Y_{P2}(\mathbf{A}_6), Y_{P2}(\mathbf{A}_8)\} > XPR_{P2}$. As a result, none of the policies in Group 1 can satisfy the maximum production limitation constraint in (10).

b) Among Group 3 policies, product $P1$ is manufactured in state 1 in policies $\mathbf{A}_{35}$, $\mathbf{A}_{36}$, $\mathbf{A}_{39}$, and $\mathbf{A}_{40}$. In these four policies, the expected production of $P1$ is smallest under policy $\mathbf{A}_{35} = [P1, M1, P2, M1]$. Similarly, product $P2$ is manufactured in state 1 in policies $\mathbf{A}_{41}$, $\mathbf{A}_{42}$, $\mathbf{A}_{45}$, and $\mathbf{A}_{46}$, and the expected production of $P2$ is smallest under policy $\mathbf{A}_{41} = [P2, M1, P1, M1]$. When $Y_{P1}(\mathbf{A}_{35}) > XPR_{P1}$ and $Y_{P2}(\mathbf{A}_{41}) > XPR_{P2}$, then no policy in Group 3 can satisfy the maximum production limitation constraint in (10). Moreover, notice that $\{Y_{P1}(\mathbf{A}_3), Y_{P1}(\mathbf{A}_4)\} > Y_{P1}(\mathbf{A}_{35}) > XPR_{P1}$ and $\{Y_{P2}(\mathbf{A}_{13}), Y_{P2}(\mathbf{A}_{14})\} > Y_{P2}(\mathbf{A}_{41}) > XPR_{P2}$, and therefore policies $\mathbf{A}_3$, $\mathbf{A}_4$, $\mathbf{A}_{13}$, and $\mathbf{A}_{14}$ of Group1 cannot be optimal. This reduces the number of potentially optimal policies to 16, and eliminates the maintenance-related critical ratios in state 2.

Proof of Proposition 10: Let us begin the proof with a monotone policy. Suppose product $P1$ is manufactured in states 1 through $j-1$, and maintenance action $M1$ is performed in states $j+2$ through N. Let us use define the following two policies: $\mathbf{A}_j = [a_1 = P1, \ldots a_{j-1} = P1, a_j = M1, \ldots, a_N = M1]$ and $\mathbf{A}_{j+1} = [a_1 = P1, \ldots, a_j = P1, a_{j+1} = M1, \ldots, \quad a_N = M1]$. When the firm considers the switch from maintenance to production in state j, the base policy is $\mathbf{A}_j$, and when it considers the switch in state $j+1$, the base policy is $\mathbf{A}_{j+1}$. The ratio of the profit earned from manufacturing $P1$, $r_{j,P1}$ (which can be defined as decreasing in j,corresponding to lower profits in deteriorated states), to the maintenance cost $c_{j,M1}$ (which can be defined as increasing in j, corresponding to higher maintenance costs in deteriorated states) is useful for the proof. Let us begin with considering the case when the ratio $\frac{r_{j,P1}}{c_{j,M1}}$ is decreasing in state j; thus, $\frac{r_{j+1,P1}}{c_{j+1,M1}} - \frac{r_{j,P1}}{c_{j,M1}} < 0$. The change in the critical ratio in (11), which can be expressed as $\gamma_{j+1}^{M1,P1} - \gamma_j^{M1,P1}$, can be compared with the change in $\frac{r_{j+1,P1}}{c_{j+1,M1}} - \frac{r_{j,P1}}{c_{j,M1}}$. Specifically, if the decrease in $\gamma_{j+1}^{M1,P1} - \gamma_j^{M1,P1}$ is higher than the decrease in $\frac{r_{j+1,P1}}{c_{j+1,M1}} - \frac{r_{j,P1}}{c_{j,M1}}$, then the ratio $\frac{r_{j,P1}}{c_{j,M1}}$ and $\gamma_j^{M1,P1}$ can intersect and cross signs only once. Therefore, it is sufficient to consider the case when

$$\left[\gamma_{j+1}^{M1,P1} - \gamma_j^{M1,P1}\right] - \left[\frac{r_{j+1,P1}}{c_{j+1,M1}} - \frac{r_{j,P1}}{c_{j,M1}}\right] \geq 0.$$

$$
\begin{aligned}
\left[\gamma_{j+1}^{M1,P1}-\gamma_{j}^{M1,P1}\right] &= \sum_{i=1}^{j}\left[-\left(\frac{r_{i,P1}}{c_{j+1,M1}}\right)\left(\frac{\widehat{\Pi}_i\left(\mathbf{A}_{j+1}\right)-\Delta_{i,j+1}^{M1,P1}\left(\mathbf{A}_{j+1}\right)}{\widehat{\Pi}_{j+1}\left(\mathbf{A}_{j+1}\right)}\right)\right] \\
&+\sum_{i=j+2}^{N}\left[\left(\frac{c_{i,M1}}{c_{j+1,M1}}\right)\left(\frac{\widehat{\Pi}_i\left(\mathbf{A}_{j+1}\right)+\Delta_{i,j+1}^{M1,P1}\left(\mathbf{A}_{j+1}\right)}{\widehat{\Pi}_{j+1}\left(\mathbf{A}_{j+1}\right)}\right)\right] \\
&+\left(\frac{EV\left(\mathbf{A}_{j+1}\right)}{c_{j+1,M1}}\right)\left\{\begin{array}{c}\sum_{i=1}^{j}\tau_{i,P1}\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})-\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)\\ +\tau_{j+1,P1}\\ +\sum_{i=j+2}^{N}\tau_{i,M1}\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})+\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)\end{array}\right\} \\
&-\sum_{i=1}^{j-1}\left[-\left(\frac{r_{i,P1}}{c_{j,M1}}\right)\left(\frac{\widehat{\Pi}_i\left(\mathbf{A}_{j}\right)-\Delta_{i,j}^{M1,P1}\left(\mathbf{A}_{j}\right)}{\widehat{\Pi}_{j}\left(\mathbf{A}_{j}\right)}\right)\right] \\
&-\sum_{i=j+1}^{N}\left[\left(\frac{c_{i,M1}}{c_{j,M1}}\right)\left(\frac{\widehat{\Pi}_i\left(\mathbf{A}_{j}\right)+\Delta_{i,j}^{M1,P1}\left(\mathbf{A}_{j}\right)}{\widehat{\Pi}_{j}\left(\mathbf{A}_{j}\right)}\right)\right] \\
&-\left(\frac{EV\left(\mathbf{A}_{j}\right)}{c_{j,M1}}\right)\left\{\begin{array}{c}\sum_{i=1}^{j-1}\tau_{i,a_i}\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j})-\Delta_{i,j}^{M1,P1}(\mathbf{A}_{j})}{\widehat{\Pi}_{j}(\mathbf{A}_{j})}\right)\\ +\tau_{j,P1}\\ +\sum_{i=j+1}^{N}\tau_{i,a_i}\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j})+\Delta_{i,j}^{M1,P1}(\mathbf{A}_{j})}{\widehat{\Pi}_{j}(\mathbf{A}_{j})}\right)\end{array}\right\}
\end{aligned}
$$

$$
\left[\gamma_{j+1}^{M1,P1}-\gamma_{j}^{M1,P1}\right]-\left[\frac{r_{j+1,P1}}{c_{j+1,M1}}-\frac{r_{j,P1}}{c_{j,M1}}\right]\geq 0 \text{ implies that}
$$

$$
\left\{\begin{array}{c}
\sum_{i=1}^{j}\left[-r_{i,P1}\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})-\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)\right]-\sum_{i=1}^{j-1}\left[-r_{i,P1}\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j})-\Delta_{i,j}^{M1,P1}(\mathbf{A}_{j})}{c_{j,M1}\widehat{\Pi}_{j}(\mathbf{A}_{j})}\right)\right] \\
+\sum_{i=j+2}^{N}\left[c_{i,M1}\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})+\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)\right]-\sum_{i=j+1}^{N}\left[c_{i,M1}\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j})+\Delta_{i,j}^{M1,P1}(\mathbf{A}_{j})}{c_{j,M1}\widehat{\Pi}_{j}(\mathbf{A}_{j})}\right)\right] \\
+EV\left(\mathbf{A}_{j+1}\right)\times\left\{\begin{array}{c}\sum_{i=1}^{j}\tau_{i,P1}\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})-\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)\\ +\tau_{j+1,P1}\\ +\sum_{i=j+2}^{N}\tau_{i,M1}\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})+\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)\end{array}\right\} \\
-EV\left(\mathbf{A}_{j}\right)\times\left\{\begin{array}{c}\sum_{i=1}^{j-1}\tau_{i,P1}\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j})-\Delta_{i,j}^{M1,P1}(\mathbf{A}_{j})}{c_{j,M1}\widehat{\Pi}_{j}(\mathbf{A}_{j})}\right)\\ +\tau_{j,P1}\\ +\sum_{i=j+1}^{N}\tau_{i,M1}\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j})+\Delta_{i,j}^{M1,P1}(\mathbf{A}_{j})}{c_{j,M1}\widehat{\Pi}_{j}(\mathbf{A}_{j})}\right)\end{array}\right\} \\
-\left[\frac{r_{j+1,P1}}{c_{j+1,M1}}\left(\frac{\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}{\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)-\frac{r_{j,P1}}{c_{j,M1}}\left(\frac{\widehat{\Pi}_{j}(\mathbf{A}_{j})}{\widehat{\Pi}_{j}(\mathbf{A}_{j})}\right)\right]
\end{array}\right\}\geq 0
$$

$$\left\{\begin{array}{c}\sum_{i=1}^{j-1}\left[\begin{array}{c}\left[\left(-r_{i,P1}+\tau_{i,P1}EV\left(\mathbf{A}_{j+1}\right)\right)\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})-\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)\right]\\-\left[\left(-r_{i,P1}+\tau_{i,P1}EV\left(\mathbf{A}_{j}\right)\right)\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j})-\Delta_{i,j}^{M1,P1}(\mathbf{A}_{j})}{c_{j,M1}\widehat{\Pi}_{j}(\mathbf{A}_{j})}\right)\right]\end{array}\right]\\+\left[\begin{array}{c}\left[\left(-r_{j,P1}+\tau_{j,P1}EV\left(\mathbf{A}_{j+1}\right)\right)\left(\frac{\widehat{\Pi}_j(\mathbf{A}_{j+1})-\Delta_{j,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)\right]\\-\left[\left(-r_{j,P1}+\tau_{j,P1}EV\left(\mathbf{A}_{j+1}\right)\right)\left(\frac{\widehat{\Pi}_j(\mathbf{A}_{j})}{c_{j,M1}\widehat{\Pi}_{j}(\mathbf{A}_{j})}\right)\right]\end{array}\right]\\+\left[\begin{array}{c}\left[\left(-r_{j+1,P1}+\tau_{j+1,P1}EV\left(\mathbf{A}_{j+1}\right)\right)\left(\frac{\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)\right]\\-\left[\left(c_{j+1,M1}+\tau_{j+1,M1}EV\left(\mathbf{A}_{j}\right)\right)\left(\frac{\widehat{\Pi}_{j+1}(\mathbf{A}_{j})+\Delta_{j+1,j}^{M1,P1}(\mathbf{A}_{j})}{c_{j,M1}\widehat{\Pi}_{j}(\mathbf{A}_{j})}\right)\right]\end{array}\right]\\+\sum_{i=j+2}^{N}\left[\begin{array}{c}\left[\left(c_{i,M1}+\tau_{i,M1}EV\left(\mathbf{A}_{j+1}\right)\right)\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})+\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)\right]\\-\left[\left(c_{i,M1}+\tau_{i,M1}EV\left(\mathbf{A}_{j}\right)\right)\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j})+\Delta_{i,j}^{M1,P1}(\mathbf{A}_{j})}{c_{j,M1}\widehat{\Pi}_{j}(\mathbf{A}_{j})}\right)\right]\end{array}\right]\end{array}\right\}\geq 0$$

The following properties can be observed from their definitions:

1. $\widehat{\Pi}_i\left(\mathbf{A}_{j+1}\right)=\widehat{\Pi}_i\left(\mathbf{A}_{j}\right)-\Delta_{i,j}^{M1,P1}\left(\mathbf{A}_{j}\right)$ for all $i<j$; thus, $\widehat{\Pi}_i\left(\mathbf{A}_{j+1}\right)<\widehat{\Pi}_i\left(\mathbf{A}_{j}\right)$ for all $i<j$.
2. $\widehat{\Pi}_i\left(\mathbf{A}_{j+1}\right)=\widehat{\Pi}_i\left(\mathbf{A}_{j}\right)$ for all $i=j$
3. $\widehat{\Pi}_i\left(\mathbf{A}_{j+1}\right)=\widehat{\Pi}_i\left(\mathbf{A}_{j}\right)+\Delta_{i,j}^{M1,P1}\left(\mathbf{A}_{j}\right)$ for all $i>j$; $\widehat{\Pi}_i\left(\mathbf{A}_{j+1}\right)>\widehat{\Pi}_i\left(\mathbf{A}_{j}\right)$ for all $i>j$.

Using the above three properties, the above inequality can be expressed as follows:

$$\left\{\begin{array}{c}\sum_{i=1}^{j}\left[\left[\left(\frac{-r_{i,P1}+\tau_{i,P1}EV(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)-\left(\frac{-r_{i,P1}+\tau_{i,P1}EV(\mathbf{A}_{j})}{c_{j,M1}\widehat{\Pi}_{j}(\mathbf{A}_{j})}\right)\right]\widehat{\Pi}_i\left(\mathbf{A}_{j+1}\right)\right]\\+\left[\left(\frac{-r_{j+1,P1}+\tau_{j+1,P1}EV(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)-\left(\frac{c_{j+1,M1}+\tau_{j+1,M1}EV(\mathbf{A}_{j})}{c_{j,M1}\widehat{\Pi}_{j}(\mathbf{A}_{j})}\right)\right]\widehat{\Pi}_{j+1}\left(\mathbf{A}_{j+1}\right)\\+\sum_{i=j+2}^{N}\left[\left[\left(\frac{c_{i,M1}+\tau_{i,M1}EV(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)-\left(\frac{c_{i,M1}+\tau_{i,M1}EV(\mathbf{A}_{j})}{c_{j,M1}\widehat{\Pi}_{j}(\mathbf{A}_{j})}\right)\right]\widehat{\Pi}_i\left(\mathbf{A}_{j+1}\right)\right]\\+\sum_{i=1}^{j}\left[\left(\frac{r_{i,P1}-\tau_{i,P1}EV(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)\Delta_{i,j+1}^{M1,P1}\left(\mathbf{A}_{j+1}\right)\right]\\+\sum_{i=j+2}^{N}\left[\left(\frac{c_{i,M1}+\tau_{i,M1}EV(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)\Delta_{i,j+1}^{M1,P1}\left(\mathbf{A}_{j+1}\right)\right]\end{array}\right\}\geq 0$$

Observe that

$$\sum_{i=1}^{j+1}\left(\frac{-r_{i,P1}+\tau_{i,P1}EV\left(\mathbf{A}_{j+1}\right)}{c_{j+1,M1}\widehat{\Pi}_{j+1}\left(\mathbf{A}_{j+1}\right)}\right)\widehat{\Pi}_i\left(\mathbf{A}_{j+1}\right)+\sum_{i=j+2}^{N}\left(\frac{c_{i,M1}+\tau_{i,M1}EV\left(\mathbf{A}_{j+1}\right)}{c_{j+1,M1}\widehat{\Pi}_{j+1}\left(\mathbf{A}_{j+1}\right)}\right)\widehat{\Pi}_i\left(\mathbf{A}_{j+1}\right)=0$$

from the definition of $EV\left(\mathbf{A}_{j+1}\right)$. Therefore, the above inequality can be simplified to the following:

$$\left\{\begin{array}{c}\sum_{i=1}^{j}\left[\left(r_{i,P1}-\tau_{i,P1}EV\left(\mathbf{A}_{j}\right)\right)\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})}{c_{j,M1}\widehat{\Pi}_{j}(\mathbf{A}_{j})}\right)\right]\\-\sum_{i=j+1}^{N}\left[\left(c_{i,M1}+\tau_{i,M1}EV\left(\mathbf{A}_{j}\right)\right)\left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})}{c_{j,M1}\widehat{\Pi}_{j}(\mathbf{A}_{j})}\right)\right]\\+\sum_{i=1}^{j}\left[\left(r_{i,P1}-\tau_{i,P1}EV\left(\mathbf{A}_{j+1}\right)\right)\left(\frac{\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)\right]\\+\sum_{i=j+2}^{N}\left[\left(c_{i,M1}+\tau_{i,M1}EV\left(\mathbf{A}_{j+1}\right)\right)\left(\frac{\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}\right)\right]\end{array}\right\}\geq 0$$

$$\left\{ \begin{array}{c} \sum_{i=1}^{j} \left[\begin{array}{c} \left[\left(r_{i,P1} - \tau_{i,P1} EV\left(\mathbf{A}_j\right) \right) \left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})}{c_{j,M1}\widehat{\Pi}_j(\mathbf{A}_j)} \right) \right] \\ + \left[\left(r_{i,P1} - \tau_{i,P1} EV\left(\mathbf{A}_{j+1}\right) \right) \left(\frac{\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})} \right) \right] \end{array} \right] \\ + \sum_{i=j+2}^{N} \left[\begin{array}{c} - \left[\left(c_{i,M1} + \tau_{i,M1} EV\left(\mathbf{A}_j\right) \right) \left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})}{c_{j,M1}\widehat{\Pi}_j(\mathbf{A}_j)} \right) \right] \\ + \left[\left(c_{i,M1} + \tau_{i,M1} EV\left(\mathbf{A}_{j+1}\right) \right) \left(\frac{\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})} \right) \right] \end{array} \right] \\ - \left[\left(c_{j+1,M1} + \tau_{j+1,M1} EV\left(\mathbf{A}_j\right) \right) \left(\frac{\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}{c_{j,M1}\widehat{\Pi}_j(\mathbf{A}_j)} \right) \right] \end{array} \right\} \geq 0$$

$$\left\{ \begin{array}{c} \left[\begin{array}{c} \sum_{i=1}^{j} \left[\left(r_{i,P1} - \tau_{i,P1} EV\left(\mathbf{A}_{j+1}\right) \right) \left(\frac{\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})} \right) \right] \\ + \sum_{i=j+2}^{N} \left[\left(c_{i,M1} + \tau_{i,M1} EV\left(\mathbf{A}_{j+1}\right) \right) \left(\frac{\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})} \right) \right] \end{array} \right] \\ + \left[\begin{array}{c} \sum_{i=1}^{j} \left[\left(r_{i,P1} - \tau_{i,P1} EV\left(\mathbf{A}_j\right) \right) \left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})}{c_{j,M1}\widehat{\Pi}_j(\mathbf{A}_j)} \right) \right] \\ - \left[\left(c_{j+1,M1} + \tau_{j+1,M1} EV\left(\mathbf{A}_j\right) \right) \left(\frac{\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}{c_{j,M1}\widehat{\Pi}_j(\mathbf{A}_j)} \right) \right] \\ + \sum_{i=j+2}^{N} \left[\left(c_{i,M1} + \tau_{i,M1} EV\left(\mathbf{A}_j\right) \right) \left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})}{c_{j,M1}\widehat{\Pi}_j(\mathbf{A}_j)} \right) \right] \end{array} \right] \end{array} \right\} \geq 0$$

Note that $\left[\begin{array}{c} \sum_{i=1}^{j} \left[\left(r_{i,P1} - \tau_{i,P1} EV\left(\mathbf{A}_{j+1}\right) \right) \left(\frac{\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})} \right) \right] \\ + \sum_{i=j+2}^{N} \left[\left(c_{i,M1} + \tau_{i,M1} EV\left(\mathbf{A}_{j+1}\right) \right) \left(\frac{\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})} \right) \right] \end{array} \right] \geq 0$ when

$$\sum_{i=1}^{j} \left[\tau_{i,P1} \Delta_{i,j+1}^{M1,P1}\left(\mathbf{A}_{j+1}\right) \right] \leq \left[\begin{array}{c} \left[\frac{\sum_{i=1}^{j} \left[r_{i,P1} \Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1}) \right] + \sum_{i=j+2}^{N} \left[c_{i,M1} \Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1}) \right]}{EV(\mathbf{A}_{j+1})} \right] \\ + \sum_{i=j+2}^{N} \left[\tau_{i,M1} \Delta_{i,j+1}^{M1,P1}\left(\mathbf{A}_{j+1}\right) \right] \end{array} \right]$$

The above condition is stated as condition 1 in (12), and can be satisfied easily. Also note that $\left[\begin{array}{c} \sum_{i=1}^{j} \left[\left(r_{i,P1} - \tau_{i,P1} EV\left(\mathbf{A}_j\right) \right) \left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})}{c_{j,M1}\widehat{\Pi}_j(\mathbf{A}_j)} \right) \right] \\ - \left[\left(c_{j+1,M1} + \tau_{j+1,M1} EV\left(\mathbf{A}_j\right) \right) \left(\frac{\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}{c_{j,M1}\widehat{\Pi}_j(\mathbf{A}_j)} \right) \right] \\ + \sum_{i=j+2}^{N} \left[\left(c_{i,M1} + \tau_{i,M1} EV\left(\mathbf{A}_j\right) \right) \left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})}{c_{j,M1}\widehat{\Pi}_j(\mathbf{A}_j)} \right) \right] \end{array} \right] \geq 0$ implies that

$$\left[\begin{array}{c} \sum_{i=1}^{j} \left[\left(r_{i,P1} - \tau_{i,P1} EV\left(\mathbf{A}_j\right) \right) \widehat{\Pi}_i\left(\mathbf{A}_{j+1}\right) \right] \\ - \left[\left(c_{j+1,M1} + \tau_{j+1,M1} EV\left(\mathbf{A}_j\right) \right) \widehat{\Pi}_{j+1}\left(\mathbf{A}_{j+1}\right) \right] \\ + \sum_{i=j+2}^{N} \left[\left(c_{i,M1} + \tau_{i,M1} EV\left(\mathbf{A}_j\right) \right) \widehat{\Pi}_i\left(\mathbf{A}_{j+1}\right) \right] \end{array} \right] \geq 0$$

Adding and subtracting the term $\left(r_{j+1,P1} - \tau_{j+1,P1} EV\left(\mathbf{A}_j\right) \right) \widehat{\Pi}_{j+1}\left(\mathbf{A}_{j+1}\right)$ into the left-hand side of the inequality provides:

$$\left[\begin{array}{c} \left[\left[\sum_{i=1}^{j} \tau_{i,P1} \widehat{\Pi}_i\left(\mathbf{A}_{j+1}\right) + \sum_{i=j+2}^{N} \tau_{i,M1} \widehat{\Pi}_i\left(\mathbf{A}_{j+1}\right) \right] \left(EV\left(\mathbf{A}_{j+1}\right) - EV\left(\mathbf{A}_j\right) \right) \right] \\ - \left[\left(r_{j+1,P1} + c_{j+1,M1} \right) + \left[\tau_{j+1,P1} EV\left(\mathbf{A}_{j+1}\right) - \tau_{j+1,M1} EV\left(\mathbf{A}_j\right) \right] \right] \widehat{\Pi}_{j+1}\left(\mathbf{A}_{j+1}\right) \end{array} \right] \geq 0.$$

The above inequality is satisfied when

$$r_{j+1,P1} + c_{j+1,M1} \leq \left[\begin{array}{c} \left[\tau_{j+1,P1} EV\left(\mathbf{A}_{j+1}\right) - \tau_{j+1,M1} EV\left(\mathbf{A}_j\right) \right] \\ + \left[\left[\sum_{i=1}^{j} \tau_{i,P1} \left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})}{\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})} \right) + \sum_{i=j+2}^{N} \tau_{i,M1} \left(\frac{\widehat{\Pi}_i(\mathbf{A}_{j+1})}{\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})} \right) \right] \left(EV\left(\mathbf{A}_{j+1}\right) - EV\left(\mathbf{A}_j\right) \right) \right] \end{array} \right]$$

This corresponds to condition 2 in (13) in the proposition. Thus, when conditions 1 and 2 are satisfied,

$$\left\{ \begin{array}{c} \left[\begin{array}{c} \sum_{i=1}^{j} \left[\left(r_{i,P1} - \tau_{i,P1} EV\left(\mathbf{A}_{j+1}\right) \right) \left(\frac{\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})} \right) \right] \\ + \sum_{i=j+2}^{N} \left[\left(c_{i,M1} + \tau_{i,M1} EV\left(\mathbf{A}_{j+1}\right) \right) \left(\frac{\Delta_{i,j+1}^{M1,P1}(\mathbf{A}_{j+1})}{c_{j+1,M1}\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})} \right) \right] \end{array} \right] \\ + \left[\begin{array}{c} \sum_{i=1}^{j} \left[\left(r_{i,P1} - \tau_{i,P1} EV\left(\mathbf{A}_{j}\right) \right) \left(\frac{\widehat{\Pi}_{i}(\mathbf{A}_{j+1})}{c_{j,M1}\widehat{\Pi}_{j}(\mathbf{A}_{j})} \right) \right] \\ - \left[\left(c_{j+1,M1} + \tau_{j+1,M1} EV\left(\mathbf{A}_{j}\right) \right) \left(\frac{\widehat{\Pi}_{j+1}(\mathbf{A}_{j+1})}{c_{j,M1}\widehat{\Pi}_{j}(\mathbf{A}_{j})} \right) \right] \\ + \sum_{i=j+2}^{N} \left[\left(c_{i,M1} + \tau_{i,M1} EV\left(\mathbf{A}_{j}\right) \right) \left(\frac{\widehat{\Pi}_{i}(\mathbf{A}_{j+1})}{c_{j,M1}\widehat{\Pi}_{j}(\mathbf{A}_{j})} \right) \right] \end{array} \right] \end{array} \right\} \geq 0$$

and $\left[\gamma_{j+1}^{M1,P1} - \gamma_{j}^{M1,P1}\right] - \left[\frac{r_{j+1,P1}}{c_{j+1,M1}} - \frac{r_{j,P1}}{c_{j,M1}}\right] \geq 0$.

Proof of Proposition 11: a) In (15), when $\delta_j^{P1,P2} < \frac{\tau_{j,P2}}{\tau_{j,P1}}$, the firm has $\left(\tau_{j,P2} - \tau_{j,P1}\delta_j^{P1,P2}\right) > 0$. Because $EV\left(\mathbf{A}_n\right) > 0$, the second term is ensured to be positive. When $\left(\frac{\tau_{j,P2} - \tau_{j,P1}\delta_j^{P1,P2}}{r_{j,P1}}\right)$ is increasing, both terms in (15) increase, resulting in increasing values of the critical ratio. b) The same approach can be used to prove the decreasing behavior of (15).

Proof of Proposition 12: In Proposition 11.a), it is shown that $\alpha_j^{P1,P2}$ is increasing when the conditions 2, 3, and 4 are satisfied. If $\frac{r_{j,P2}}{r_{j,P1}}$ is increasing in j, as the first condition states, the relative values of $\frac{r_{j,P2}}{r_{j,P1}}$ and $\alpha_j^{P1,P2}$ can switch only once. Specifically, if $\frac{r_{1,P2}}{r_{1,P1}} > \alpha_1^{P1,P2}$, product $P2$ is preferred. As $\frac{r_{j,P2}}{r_{j,P1}}$ decreases in state j the firm can have $\frac{r_{j,P2}}{r_{j,P1}} < \alpha_j^{P1,P2}$ in a state j, and switch to manufacturing $P1$. Because of the decreasing behavior of $\frac{r_{j,P2}}{r_{j,P1}}$, the firm would have $\frac{r_{i,P2}}{r_{i,P1}} < \alpha_i^{P1,P2}$ in all states $i > j$.

Proof of Proposition 13: a) In (16), when $\delta_j^{M1,M2} > \frac{\tau_{j,M2}}{\tau_{j,M1}}$, the firm has $\left(-\tau_{j,M2} + \tau_{j,M1}\delta_j^{M1,M2}\right) > 0$. Because $EV\left(\mathbf{A}_n\right) > 0$, the second term is ensured to be positive. When $\left(\frac{-\tau_{j,M2} + \tau_{j,M1}\delta_j^{M1,M2}}{c_{j,M1}}\right)$ is increasing, both terms in (16) increase, resulting in increasing values of the critical ratio. The same approach can be used to prove the decreasing behavior of (16).

Proof of Proposition 14: In Proposition 13.a), it is shown that $\lambda_j^{M1,M2}$ is increasing when the conditions 2, 3, and 4 are satisfied. If $\frac{c_{j,M2}}{c_{j,M1}}$ is decreasing in j, as the first condition states, the relative values of $\frac{c_{j,M2}}{c_{j,M1}}$ and $\lambda_j^{M1,M2}$ can switch only once. Specifically, if $\frac{c_{2,M2}}{c_{2,M1}} > \lambda_j^{M1,M2}$, maintenance action $M1$ is preferred. As $\frac{c_{j,M2}}{c_{j,M1}}$ decreases in state j the firm can have $\frac{c_{j,M2}}{c_{j,M1}} < \lambda_j^{M1,M2}$ in a state j and switch to maintenance action $M2$. Because of the decreasing behavior of $\frac{c_{2,M2}}{c_{2,M1}}$, the firm would have $\frac{c_{j,M2}}{c_{j,M1}} < \lambda_j^{M1,M2}$ in all states $i > j$.

Example 1: This example highlights the impact of the minimum production requirements on the optimal policy choice and the critical ratios used in determining the optimal solution. Consider the scenario in which the two products earn a per unit profit of $\rho_{P1} = \$15$ and $\rho_{P2} = \$45$, respectively. As the process deteriorates, the number of defective units produced increases. The yield for $P1$ in various states is $y_{1,P1} = 100$, $y_{2,P1} = 90$, $y_{3,P1} = 80$, and for $P2$ it is $y_{1,P2} = 95$, $y_{2,P2} = 65$, $y_{3,P2} = 60$. The revenue earned for each product in each state is described as $r_{i,a_i} = \rho_{P1} \times y_{i,a_i}$, and therefore, $r_{1,P1} = \$1,500$, $r_{2,P1} = \$1,350$, $r_{3,P1} = \$1,200$ for $P1$ and $r_{1,P2} = \$4,275$, $r_{2,P2} = \$2,925$, $r_{3,P2} = \$2,700$ for $P2$. The expected processing times for each product are $\tau_{i,P1} = 1$ and $\tau_{i,P2} = 2$ in states $i = 1,2,3$. One of the trade-offs between the two products is that $P2$ takes a longer expected time to produce but brings a higher profit. The maintenance costs are as follows: $c_{i,M1} = \$1,300$ for a minor maintenance action and $c_{i,M2} = \$2,600$ for a major maintenance activity in states $i = 2,3,4$. The expected time for each maintenance action is $\tau_{i,M1} = 1$ and $\tau_{i,M2} = 2$ in states $i = 2,3,4$. Therefore, a major maintenance action is twice long in its expected processing time and twice expensive compared with a minor maintenance action. The probability transition matrix describing the deterioration from manufacturing actions and the improvement from maintenance actions are as follows:

$$[p_{ij}^{P1}] = \begin{bmatrix} 0.70 & 0.20 & 0.05 & 0.05 \\ 0 & 0.70 & 0.25 & 0.05 \\ 0 & 0 & 0.70 & 0.30 \\ 0 & 0 & 0 & 1.00 \end{bmatrix} \qquad [p_{ij}^{P2}] = \begin{bmatrix} 0.50 & 0.20 & 0.20 & 0.10 \\ 0 & 0.50 & 0.25 & 0.25 \\ 0 & 0 & 0.50 & 0.50 \\ 0 & 0 & 0 & 1.00 \end{bmatrix}$$

$$[p_{ij}^{M1}] = \begin{bmatrix} 1.00 & 0 & 0 & 0 \\ 0.50 & 0.50 & 0 & 0 \\ 0.25 & 0.25 & 0.50 & 0 \\ 0.10 & 0.20 & 0.20 & 0.50 \end{bmatrix} \qquad [p_{ij}^{M2}] = \begin{bmatrix} 1.00 & 0 & 0 & 0 \\ 0.75 & 0.25 & 0 & 0 \\ 0.35 & 0.35 & 0.30 & 0 \\ 0.30 & 0.20 & 0.20 & 0.30 \end{bmatrix}.$$

If the decision maker ignores the minimum production constraints in (9), then all 64 policies listed in Table 1 are candidates for the optimal solution. It can be verified that in the first three states that the ratio of profits exceed the production-related critical ratios (i.e., $\frac{r_{i,P2}}{r_{i,P1}} > \alpha_i^{P1,P2}$ for $i = 1, 2, 3$) recommending the manufacturing of product $P2$, and the ratio of maintenance costs exceeds the maintenance-related critical ratio (i.e., $\frac{c_{4,M2}}{c_{4,M1}} > \lambda_4^{M1,M2}$) employing minor maintenance in the worst state. As a result, in the absence of minimum production requirements, the optimal policy is $\mathbf{A}_{15} = [P2, P2, P2, M1]$ with the expected profit of $EV(\mathbf{A}_{15}) = \$763.95$. According to $\mathbf{A}_{15}$, however, the firm should manufacture only product $P2$ with the expected production of $\mathbf{Y}_{P2}(\mathbf{A}_{15}) = 24.579$, without any production of $P1$, i.e., $\mathbf{Y}_{P1}(\mathbf{A}_{15}) = 0$.

The firm may have prior commitments to its customers, however, so it might be necessary to produce both products. Let us now focus on the 28 policies listed in Table 2, but continue to ignore the minimum production requirements in (9). In this variant of the problem, the least costly switch takes place in the second state, therefore, policy $\mathbf{A}_{11} = [P2, P1, P2, M1]$ becomes the new optimal solution with its expected profit of $EV(\mathbf{A}_{11}) = \$760.35$. The change in the policy is due to the fact the firm's production yield of $P2$ in state 2 shows a significant drop (from 95 in state 1 to 65 in state 2). As a result, $P2$ provides a relatively lower profit in proportion than it does in other states compared with the profits generated from $P1$. The expected production per unit time from policy $\mathbf{A}_{11}$ is $\mathbf{Y}_{P1}(\mathbf{A}_{11}) = 18.182$ units of $P1$ and $\mathbf{Y}_{P2}(\mathbf{A}_{11}) = 18.131$ units of $P2$, and are both positive.

Let us now incorporate the minimum production requirements into the problem. Semiconductor manufacturers typically have significant commitments to provide downstream electronics manufacturers with their mature products such as $P1$. For standard-technology products, the market is established better than the high-end products. So, our example features a higher requirement for $P1$ than for $P2$. A similar example can be produced for the opposite case without changing the structural results.

The example considers the minimum production requirements of $MPR_{P1} = 36$ units of $P1$ and $MPR_{P2} = 6$ units of $P2$ per unit time for the constraint in (9). There are only three policies that satisfy this constraint: $\mathbf{A}_6 = [P1, P2, P1, M2]$, $\mathbf{A}_{10} = [P2, P1, P1, M2]$ and $\mathbf{A}_{35} = [P1, M1, P2, M1]$. The optimal policy under (9) is $\mathbf{A}_{10} = [P2, P1, P1, M2]$ with its expected profit of $EV(\mathbf{A}_{10}) = \$464.64$. Compared with the optimal policy in the absence of a minimum production requirement in (9), the firm loses 39.18% (=(\$763.95-464.64)/\$763.95 x 100%) of its profits from $\mathbf{A}_{15}$ (the pure product policy of $P2$) and 38.89% (=(\$760.35-464.64)/\$763.95 x 100%) of its profits from $\mathbf{A}_{11}$. Expected production under policy $\mathbf{A}_{10}$ is $\mathbf{Y}_{P1}(\mathbf{A}_{10}) = 40.571$ units of $P1$ and $\mathbf{Y}_{P2}(\mathbf{A}_{10}) = 9.161$ units of $P2$. Note that $\mathbf{Y}_{P1}(\mathbf{A}_{10}) = 40.571 > MPR_{P1} = 36$ and $\mathbf{Y}_{P2}(\mathbf{A}_{10}) = 9.161 > MPR_{P2} = 6$. It can be concluded that satisfying these minimum production requirements costs the firm approximately 39% of its profits in this example. Because of its bigger requirement, the firm has to produce $P1$ more frequently under (9). This is accomplished in state 3 based on a similar reason. Once again, the drop in the yield of product $P2$ in state 3 (60 units compared with 95 in state 1) makes the choice of producing $P1$ in state 3 a better alternative than other states. Moreover, the optimal maintenance decision in state 4 is $M2$. Utilizing the minor maintenance action $M1$ reduces the expected production to $5.182 < 6$ and making (9) infeasible. This exemplifies how the critical ratio $\lambda_4^{M1,M2}$ can be eliminated from consideration due to minimum production requirements in (9).

Propositions 5 and 6 have already established the conditions for increasing the throughput of both products. In these propositions, it is shown that maintenance can play a strategic role in increasing in the expected production of a product. The above comparison highlights the impact of replacing a maintenance action with a production action in a deteriorated state in order to increase the throughput of a high demand product. In the next example, we demonstrate how the firm can increase its expected production by employing a major maintenance action, rather than minor maintenance. Consider policy $\mathbf{A}_{21} = [P1, P2, M1, M1]$ with its expected production of $\mathbf{Y}_{P1}(\mathbf{A}_{21}) = 20.548 < MPR_{P1} = 36$ and $\mathbf{Y}_{P2}(\mathbf{A}_{21}) = 15.137 > MPR_{P2} = 6$. We next show how switching to a major maintenance action from a minor maintenance action can increase the throughput. Consider now policy $\mathbf{A}_{24} = [P1, P2, M2, M2]$ with same set of production decisions but with a major maintenance action in states 3 and 4. Policy $\mathbf{A}_{24}$ leads to the expected production of $\mathbf{Y}_{P1}(\mathbf{A}_{24}) = 21.428 < MPR_{P1} = 36$ and $\mathbf{Y}_{P2}(\mathbf{A}_{24}) = 12.809.137 < MPR_{P2} = 6$. While the throughput for $P1$ increases with this policy, throughput for $P2$ decreases. The reason for a small increase in $P1$ and a reduction in $P2$ is due to the long expected processing time of maintenance action $M2$. While employing a major maintenance action can increase the throughput of a product, it cannot always guarantee an increase due to possibly lengthy expected processing times.

The firm can also increase throughput by moving its maintenance to an earlier state. This can be demonstrated by comparing policies $\mathbf{A}_{21} = [P1, P2, M1, M1]$ and $\mathbf{A}_{35} = [P1, M1, P2, M1]$. While both policies have the same number of production and maintenance actions, manufacturing of $P2$ moves from state 2 to 3 and maintenance $M1$ moves from state 3 to state 2. Recall that policy $\mathbf{A}_{21} = [P1, P2, M1, M1]$ brings an expected production of $\mathbf{Y}_{P1}(\mathbf{A}_{21}) = 20.548 < MPR_{P1} = 36$ and $\mathbf{Y}_{P2}(\mathbf{A}_{21}) = 15.137 > MPR_{P2} = 6$, and therefore, is an infeasible policy. Policy $\mathbf{A}_{35} = [P1, M1, P2, M1]$, however, leads to the expected production of $\mathbf{Y}_{P1}(\mathbf{A}_{35}) = 42.857 > MPR_{P1} = 36$ and $\mathbf{Y}_{P2}(\mathbf{A}_{35}) = MPR_{P2} = 6$, and satisfies both of the minimum production requirement constraints. Policy $\mathbf{A}_{35}$ increases the throughput of $P1$ manufactured in state 1 by increasing the frequency of the process being in this state, i.e., by increasing the steady-state probability. While this policy reduces the throughput for $P2$, it makes both products satisfy the minimum production requirement in (9). This example shows that the firm can increase the throughput of its products manufactured in better states by moving its maintenance action to an earlier state, i.e., before the process gets extremely deteriorated. Thus, it is not just the *frequency* of maintenance that influences the firm's throughput; the *timing* of maintenance is as influential in the system's throughput. In conclusion, through this example, we have demonstrated that the firm can increase the throughput of its products by 1) switching from maintenance to production actions in deteriorated states, 2) employing a major maintenance action rather than minor maintenance, 3) moving its maintenance actions to better states. All three alternatives rely on the changes in the expected processing times.

Let us next present the effectiveness of Proposition 7 under the minimum production requirements in (9). The expected production for products $P1$ and $P2$ using policy $\mathbf{A}_{42}$ is $\mathbf{Y}_{P1}(\mathbf{A}_{42}) = 17.391 < MPR_{P1} = 36$ and $\mathbf{Y}_{P2}(\mathbf{A}_{42}) = 19.361 > MPR_{P2} = 6$. According to part a) of Proposition 7, the two conditions are not satisfied simultaneously, and therefore, Group 3 policies cannot be eliminated from consideration. This is true because policy $\mathbf{A}_{35} = [P1, M1, P2, M1]$ is a Group 3 policy and is one of the candidate solutions. Next, we investigate part b) of Proposition 7. Note that $\mathbf{Y}_{P1}(\mathbf{A}_{24}) = 21.420 < MPR_{P1} = 36$ and $\mathbf{Y}_{P2}(\mathbf{A}_{28}) = 13.238 > MPR_{P2} = 6$. These imply that the conditions in part b) of the proposition are met, and no policy in Group 2 can be optimal as reflected in the set of potentially optimal policies. The consequence of this result is the following: The firm *must* produce in state 3. As a result, the critical ratios $\gamma_3^{M1,P1}$ and $\lambda_3^{M1,M2}$ for state 3 can be eliminated from consideration, leaving the comparison to only $\alpha_3^{P1,P2}$. Parts c) and d) of the proposition utilize the same conditions: $\mathbf{Y}_{P1}(\mathbf{A}_{40}) = 41.67 > MPR_{P1} = 36$ and $\mathbf{Y}_{P2}(\mathbf{A}_{46}) = 18.855 > MPR_{P2} = 6$. These imply that conditions in part c) are not met, so not all policies from Groups 2 and 3 can be eliminated. Once again, policy $\mathbf{A}_{35} = [P1, M1, P2, M1]$ is a Group 3 policy and is one of the candidate solutions, confirming the conclusion of the proposition. Part d) has the same conclusion as the firm cannot eliminate Group 1 policies as two of the three potentially optimal policies, $\mathbf{A}_6 = [P1, P2, P1, M2]$ and $\mathbf{A}_{10} = [P2, P1, P1, M2]$ belong to Group 1. In sum, the conclusions from Proposition 7 are consistent with our findings. Moreover, from a critical ratio perspective, the firm has to utilize critical ratios $\alpha_1^{P1,P2}$ in state 1, $\alpha_2^{P1,P2}$, $\gamma_2^{M1,P1}$, $\lambda_2^{M1,M2}$ in state 2, $\alpha_3^{P1,P2}$ in state 3. With this example, we have demonstrated how the minimum production requirements influence the critical ratios and how they help the decision maker reduce the number of comparisons in determining the optimal policy.

Example 2: To highlight the monotonicity conditions, this example is framed as a three-state problem. The firm's profit from manufacturing $P1$ and $P2$ in states 1 and 2 are $[r_{1,P1}, r_{2,P1}] = [\$1000, \$750]$ and $[r_{1,P2}, r_{2,P2}] = [\$1500, \$1000]$. Notice that the profit for each manufacturing action is decreasing in state. The maintenance cost of $M1$ and $M2$ in states 2 and 3 are as follows: $[c_{2,M1}, c_{3,M1}] = [\$625, \$780]$ and $[c_{2,M2}, c_{3,M2}] = [\$1000, \$1000]$. The expected processing times of these four actions do not change with state: $[\tau_{i,P1}, \tau_{i,P2}, \tau_{i,M1}, \tau_{i,M2}] = [1, 2.5, 1, 2]$ for all i. The machine state transition probabilities for the production actions are:

$$[p_{ij}^{P1}] = \begin{bmatrix} 0.75 & 0.05 & 0.2 \\ 0 & 0.7 & 0.3 \\ 0 & 0 & 1 \end{bmatrix} \qquad [p_{ij}^{P2}] = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0 & 0.4 & 0.6 \\ 0 & 0 & 1 \end{bmatrix},$$

and the machine state transition probabilities for the maintenance actions are:

$$[p_{ij}^{M1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0.05 & 0.45 & 0.5 \end{bmatrix} \qquad [p_{ij}^{M2}] = \begin{bmatrix} 1 & 0 & 0 \\ 0.8 & 0.2 & 0 \\ 0.25 & 0.5 & 0.25 \end{bmatrix}.$$

Note that the ratio of deterioration probabilities in states 1 and 2 are: $\delta_1^{P1,P2} = \frac{1-p_{11}^{P2}}{1-p_{11}^{P1}} = \frac{0.50}{0.25} = 2.0$ and $\delta_2^{P1,P2} = \frac{1-p_{22}^{P2}}{1-p_{22}^{P1}} = \frac{0.60}{0.30} = 2.0$. First, we demonstrate how our critical ratios can determine the optimal policy. Let us consider the base policy $\mathbf{A}_5 = [P1, P1, M2]$ with its expected value $EV(\mathbf{A}_5) = \$284.50$. We first calculate the production-related critical ratio in state 2: $\alpha_2^{P1,P2} = \delta_2^{P1,P2} + \left(\frac{EV(\mathbf{A}_5)}{r_{2,P1}}\right)\left(\tau_{2,P2} - \tau_{2,P1}\delta_2^{P1,P2}\right) = 2 + \left(\frac{\$284.50}{\$750}\right)(2.5 - 1 \times 2) = 2.19$. The ratio of production profits in state 2 is lower than the critical ratio, i.e., $\frac{r_{2,P2}}{r_{2,P1}} = \frac{\$1000}{\$750} = 1.33 < \alpha_2^{P1,P2} = 2.19$, and therefore, $a_2^* = P1$. Since P1 is preferred in state 2, we continue to use policy $\mathbf{A}_5 = [P1, P1, M2]$ as the reference policy. The production-related critical ratio for state 1 is: $\alpha_1^{P1,P2} = \delta_1^{P1,P2} + \left(\frac{EV(\mathbf{A}_5)}{r_{1,P1}}\right)\left(\tau_{1,P2} - \tau_{1,P1}\delta_1^{P1,P2}\right) = 2 + \left(\frac{\$284.50}{\$1000}\right)(2.5 - 1 \times 2) = 2.14$. The ratio of production profits in state 1 is lower than the critical ratio, i.e., $\frac{r_{1,P2}}{r_{1,P1}} = \frac{\$1500}{\$1000} = 1.5 < \alpha_1^{P1,P2} = 2.14$, and therefore, $a_1^* = P1$. As a result, policy $\mathbf{A}_5 = [P1, P1, M2]$ is the optimal solution to this problem. Although the optimal policy is monotone, the conditions reported in Sloan (2008) are not met. Specifically, condition C3′, corresponding to the reward rate superadditivity property, and condition C5′, corresponding to the holding time subadditivity property, are both violated. For the former condition, note that $\frac{r_{1,P1}}{(1-p_{11}^{P1})} - \frac{r_{1,P2}}{(1-p_{11}^{P2})} = \frac{1000}{0.25} - \frac{1500}{.5} = 1000$, while $\frac{r_{2,P1}}{(1-p_{22}^{P1})} - \frac{r_{2,P2}}{(1-p_{22}^{P2})} = \frac{750}{0.3} - \frac{1000}{.6} = 833.33$. This decrease violates condition C3′. For the latter condition, note that $\frac{\tau_{1,P1}}{(1-p_{11}^{P1})} - \frac{\tau_{1,P2}}{(1-p_{11}^{P2})} = \frac{1}{0.25} - \frac{2.5}{.5} = -1$, while $\frac{\tau_{2,P1}}{(1-p_{22}^{P1})} - \frac{\tau_{2,P2}}{(1-p_{22}^{P2})} = \frac{1}{0.3} - \frac{2.5}{.6} = -0.833$. This increase violates condition C4′. In contrast, the sufficient conditions in Proposition 12 are satisfied for this problem. Condition 1 is met as the ratio of production profits is decreasing in state: $\frac{r_{1,P2}}{r_{1,P1}} = \frac{\$1500}{\$1000} = 1.5 > \frac{r_{2,P2}}{r_{2,P1}} = \frac{\$750}{\$1000} = 1.33$. Condition 2 is satisfied because the ratio of deterioration probabilities is non-decreasing in state: $\delta_1^{P1,P2} = 2 \geq \delta_2^{P1,P2} = 2$. The problem also meets Condition 3 because $\frac{\tau_{1,P2}}{\tau_{1,P1}} = \frac{2.5}{1} = 2.5 > \delta_1^{P1,P2} = 2$ in state 1 and $\frac{\tau_{2,P2}}{\tau_{2,P1}} = 2.5 > \delta_2^{P1,P2} = 2$ in state 2. Condition 4 is also satisfied as $\frac{\tau_{j,P2} - \tau_{j,P1}\delta_j^{P1,P2}}{r_{j,P1}}$ is increasing in j: $\frac{\tau_{1,P2} - \tau_{1,P1}\delta_1^{P1,P2}}{r_{1,P1}} = \frac{2.5 - 1 \times 2}{1000} = 0.0005 < \frac{\tau_{2,P2} - \tau_{2,P1}\delta_2^{P1,P2}}{r_{2,P1}} = \frac{2.5 - 1 \times 2}{750} = 0.00067$. As a result, all four conditions of Proposition 12 are satisfied ensuring that the optimal policy is monotone, whereas the conditions in Sloan (2008) are not met. In addition, we have demonstrated how the critical ratios are used to determine the optimal solution.

Example 3: To highlight the monotonicity conditions, this example is framed as a three-state problem. The firm's profit from manufacturing $P1$ and $P2$ in states 1 and 2 are $[r_{1,P1}, r_{2,P1}] = [\$350, \$120]$ and $[r_{1,P2}, r_{2,P2}] = [\$500, \$100]$. Notice that the profit for each manufacturing action is decreasing in state. The maintenance cost of $M1$ and $M2$ in states 2 and 3 are as follows: $[c_{2,M1}, c_{3,M1}] = [\$200, \$300]$ and $[c_{2,M2}, c_{3,M2}] = [\$375, \$375]$. Note that the maintenance cost for $M1$ is increasing in state whereas it remains the same for $M2$. The processing times of these four actions do not change with state and are equal to: $[\tau_{i,P1}, \tau_{i,P2}, \tau_{i,M1}, \tau_{i,M2}] = [1, 2, 1, 2]$. The machine state transition probabilities for the production actions are:

$$\left[p_{ij}^{P1}\right] = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left[p_{ij}^{P2}\right] = \begin{bmatrix} 0.25 & 0.375 & 0.375 \\ 0 & 0.25 & 0.75 \\ 0 & 0 & 1 \end{bmatrix},$$

and the machine state transition probabilities for the maintenance actions are:

$$\left[p_{ij}^{M1}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0.1 & 0.5 & 0.4 \end{bmatrix} \qquad \left[p_{ij}^{M2}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0.9 & 0.1 & 0 \\ 0.39 & 0.51 & 0.1 \end{bmatrix}.$$

Note that the ratio of improvement probabilities in states 2 and 3 are: $\delta_2^{M1,M2} = \frac{1-p_{22}^{M2}}{1-p_{22}^{M1}} = \frac{0.90}{0.50} = 1.80$ and $\delta_3^{M1,M2} = \frac{1-p_{33}^{M2}}{1-p_{33}^{M1}} = \frac{0.90}{0.60} = 1.50$. First, we demonstrate how our critical ratios can determine the optimal policy. Let us consider the base policy $\mathbf{A}_9 = [P1, M1, M1]$ with its expected value $EV(\mathbf{A}_9) = \$17.90$. We first calculate the maintenance-related critical ratio in state 3: $\lambda_3^{M1,M2} = \delta_3^{M1,M2} + \left(\frac{EV(\mathbf{A}_9)}{c_{3,M1}}\right)\left(-\tau_{3,M2} + \tau_{3,M1}\delta_3^{M1,M2}\right) = 1.50 + \left(\frac{\$17.90}{\$300}\right)(-2 + 1 \times 1.50) = 1.47$. The ratio of maintenance costs in state 3 is lower than the critical ratio, i.e., $\frac{c_{3,M2}}{c_{3,M1}} = \frac{\$375}{\$300} = 1.25 < \lambda_3^{M1,M2} = 1.47$, and therefore, $a_3^* = M2$. Now that M2 is preferred in state 3, we have the policy $\mathbf{A}_{10} = [P1, M1, M2]$ with its expected value $EV(\mathbf{A}_{10}) = \$38.10$. In state 2, we compare this policy with policy $\mathbf{A}_{15} = [P1, M2, M2]$. The maintenance-related critical ratio for state 2 is: $\lambda_2^{M1,M2} = \delta_2^{M1,M2} + \left(\frac{EV(\mathbf{A}_{10})}{c_{2,M1}}\right)\left(-\tau_{2,M2} + \tau_{2,M1}\delta_2^{M1,M2}\right) = 1.80 + \left(\frac{\$38.10}{\$200}\right)(-2 + 1 \times 1.80) = 1.76$. The ratio of maintenance costs in state 2 is greater than the critical ratio, i.e., $\frac{c_{2,M2}}{c_{2,M1}} = \frac{\$375}{\$200} = 1.875 > \lambda_2^{M1,M2} = 1.76$, and therefore, $a_2^* = M1$. Similarly, we can check the best action in state 1 by comparing policy $\mathbf{A}_{10} = [P1, M1, M2]$ with $\mathbf{A}_{14} = [P2, M1, M2]$. The comparison of the ratio of profits with the critical ratio of state 1 reveals that the optimal action is $P1$. As a result, policy $\mathbf{A}_{10} = [P1, M1, M2]$ is the optimal solution to this problem. Expected values of all policies are provided in the Appendix where the monotone policy $\mathbf{A}_{10}$ is proven to be the optimal policy. Although the optimal policy is monotone, the conditions reported in Sloan (2008) are not met. Specifically, Condition 4, corresponding to the subadditive property of the transition probabilities, is violated. For $l = 2$ and $i = 2$, $\sum_{j=l}^{N}\left[p_{ij}^{M2} - p_{ij}^{M1}\right] = -0.4$, while for $l = 2$ and $i = 3$, $\sum_{j=l}^{N}\left[p_{ij}^{M2} - p_{ij}^{M1}\right] = -0.29$, an increase, which violates condition C4. In contrast, the sufficient conditions in Proposition 14 are satisfied for this problem. Condition 1 is met as the ratio of maintenance costs is decreasing in state: $\frac{c_{2,M2}}{c_{2,M1}} = \frac{\$375}{\$200} = 1.875 > \frac{c_{2,M2}}{c_{2,M1}} = \frac{\$375}{\$300} = 1.25$. Condition 2 is satisfied because the ratio of improvement probabilities is decreasing in state: $\delta_2^{M1,M2} = 1.80 > \delta_3^{M1,M2} = 1.50$. The problem complies with Condition 3 because $\frac{\tau_{2,M2}}{\tau_{2,M1}} = \frac{2}{1} = 2 > \delta_2^{M1,M2} = 1.80$ in state 2 and $\frac{\tau_{3,M2}}{\tau_{3,M1}} = 2 > \delta_3^{M1,M2} = 1.50$ in state 3. Condition 4 is also satisfied as $\frac{-\tau_{j,M2}+\tau_{j,M1}\delta_j^{M1,M2}}{c_{j,M2}}$ is decreasing in j: $\frac{-\tau_{2,M2}+\tau_{2,M1}\delta_2^{M1,M2}}{c_{2,M2}} = \frac{-2+1\times1.80}{200} = -0.001 > \frac{-\tau_{3,M2}+\tau_{3,M1}\delta_3^{M1,M2}}{c_{3,M2}} = \frac{-2+1\times1.50}{300} = -0.00167$. As a result, all four conditions of Proposition 14 are satisfied ensuring that the optimal policy is monotone, while the conditions in Sloan (2008) failed to do so. Moreover, it is shown that the critical ratios enable the decision maker to determine the optimal solution.