Satisfying Warranty Claims on an Obsolete Product

Dwayne Cole
College of Business Administration
Florida A&M University
Tallahassee, FL 32301
dwayne.cole@famu.edu

Burak Kazaz
Martin J. Whitman School of Management
Syracuse University
Syracuse, NY 13244
bkazaz@syr.edu

Scott Webster
W.P. Carey School of Business
Arizona State University
Tempe, AZ 85287
scott.webster@asu.edu

ABSTRACT

We investigate a problem faced by a durable-goods manufacturer of a product that is no longer manufactured but still under warranty. A supplier announces that a component of the product will be phased out and specifies a deadline for the final order. A common response in traditional practice is to place a final order sufficient to cover future warranty claims. We analyze and compare this policy with a policy that uses a trade-in program to supplement the final order quantity.

KEYWORDS: Warranty, trade-in policies, sustainability

INTRODUCTION

We investigate a problem faced by a durable-goods manufacturer of a product that is no longer manufactured but still under warranty. A supplier announces that a component of the product will be phased out and specifies a deadline for the final order. The manufacturer projects the component needs for the product under warranty and considers a two-stage decision problem: (1) the size of the final order and, in the event that the final order is less than actual requirements, (2) the design of a trade-in program for component harvesting.

The importance and prevalence of this problem have increased over time due to shrinking product life-cycles and growth in outsourcing. These trends are especially pronounced in the computer industry where the high pace of change and technical challenges favor supply chains of independent firms with specialized expertise (e.g., AMD and Intel for processors, Seagate and Western Digital and for hard drives, Cisco and D-Link for routers, Flextronics and Selectron for assembly).

We consider the setting where the component phase-out announcement (CPOA) occurs after the manufacturer has discontinued manufacturing and sales of the parent product. The particular component contributes significantly to the value of the product and is not easily or inexpensively obtained from alternative suppliers (e.g., highly engineered and expensive component). These features elevate the importance of managerial attention on an effective response to the CPOA. We investigate how a firm's optimal final order quantity and trade-in program decisions are influenced by industry and market characteristics.

Our main contribution lies in two observations that come from our analysis. First, a trade-in program has potential to significantly reduce a firm's warranty liability. Second, there are two key indicators that savings from a trade-in program will be significant. One indicator is the difference between the component cost and the marginal cost of the first unit acquired via trade-in. A second indicator is the expected fraction of products under warranty that will fail. Both of these values should not be difficult for a firm to estimate.

ELEMENTS OF THE COMPONENT PHASE-OUT ANNOUNCEMENT PROBLEM

As a new generation of a component is introduced and the volume of the previous generation declines, a supplier eventually ceases to supply the older generation component and announces a time-line for phase out. While it is possible that a CPOA may occur when the manufacturer is still producing a product with the component, we limit consideration to the case where the product is no longer being manufactured (as is consistent with CPOA timing examples described to us by those in industry). Thus, the final component purchase decision is driven by warranty obligation considerations. Durable-goods manufacturers commonly offer a limited-time warranty to consumers.

The CPOA problem can be viewed as a two-stage decision problem. The first-stage decision is the number of components in the final order. After the final order is placed, component demand is realized over time. The second-stage decision, if necessary, is the price discount to be offered on a trade-in.

We assume that the firm has access to customer-specific warranty data. In these settings, the trade-in offer can be targeted to specific customers based on product age and time remaining under warranty. We also assume that warranty claim demand and trade-in return volume are known with certainty. Through this simplifying assumption we are able to gain some insight into what drives the value of a trade-in program. We leave consideration of uncertainty for future research.

LITERATURE REVIEW

Most consumer durables come with either a pro-rata refund or a free repair/replacement warranty policy (Blischke and Murthy 1992). Murthy et al. (2004) provide a comprehensive review of various issues associated with warranty management. Warranty claims are driven by the warranty population, usage characteristics, product reliability, and warranty terms. Seitz (2007) reports that the use of recovered components to satisfy warranty claims is a common practice in the automobile and home appliance industries. Cisco began using returns to support warranty claims in 2008. The initiative increased the recovered value from returns by nine-fold, from 5% to 45% (Nidumolu et al. 2009).

There are three streams of research related to our two-stage problem of how a manufacturer determines the size of the final order and designs the trade-in program for component harvesting. One of the streams relates to the final order quantity problem, which is the first-stage decision in our model. Fortuin (1980) introduces a model wherein the machines' remaining operating life is divided into discrete intervals and the number of components that fail in each interval is random. He proposes a method for estimating the component stock-out probability as a function of the final order quantity. Teunter and Hansveld (1998) extend the single component setting of the previous papers to a multi-component ordering problem. They show the multi-component problem can be decomposed into independent single-component ordering problems. Bradley and Guerrero (2009) also consider the multi-component final buy problem, though in contrast to Teunter and Hansveld (1998) who assume all components are phased out at the same moment in time, components are phased out gradually over time. Teunter and Fortuin (1999) develop a single-stage dynamic program to determine the optimal final order quantity for a single component. Their model allows for the possibility of harvesting components from returns as a source for spare-parts. However, the firm passively accepts used-product returns. This differs from our model where their firm actively manages return volumes through the setting of trade-in discounts over time. In sum, our work extends the literature on the final order quantity problem by supplementing the final order quantity with product acquired via a trade-in program—the second stage in our model.

A second related stream of research addresses the use of returns as a source for spare parts. The literature on this topic is vast (e.g., see Kennedy et al. 2002 for a review). Within this literature, a number of researchers have studied the problem of managing component parts after the parent product has reached the end of its sales life-cycle. Minner and Kleber (2001) examine a setting where the firm produces a new component and remanufactures used components to meet a deterministic demand for spare parts. The authors develop a dynamic inventory framework and use optimal control techniques to determine an optimal production and recovery strategy for a firm that passively accepts used products. Spengler and Schröter (2003) also develop a dynamic model that integrates component harvesting. They use the model to study flows of new product sales, spare parts demands, product returns, and recovery rates. Their model focuses on the behavior of the spare parts management system. Acquisition cost and returns flows are exogenous in their model, and are endogenous in our model; they are influenced by the firm's choices. Inderfurth and Mukherjee (2008) develop a decision model where the firm must meet a dynamic demand for spare-parts during the phase-out period. Our model resembles that of Inderfurth and Mukherjee (2008) in the sense that the firm chooses the final order quantity and remanufactures components from returned products to meet dynamic demand for spare parts. However, our model improves their model in two ways: (1) As opposed to an exogenous flow of returns, we assume that the firm proactively acquires used products from its install base; the firm sets trade-in discounts to influence the timing and quantity of the return flow. (2) We account for the impact of returns of product under warranty on future warranty claims.

A third stream of related literature examines the relationships between new product prices, trade-in rebates, product return volumes, and new product purchases. Ray et al. (2005) examine how a trade-in program for a product that is remanufactured can be used as a price-discrimination mechanism to increase profits. Bruce et al. (2006) study trade-in programs for expensive durables purchased with the aid of a loan (e.g., automobiles). They examine the relationship between the magnitude of the trade-in discount and the durability of the product. Rao et al. (2009) study the value of trade-in programs for products in which used product prices are negatively affected due to information asymmetry (e.g., a positive probability of buying a "lemon"). A key difference between these papers and our work is that there is no demand for used components that must be met, as is the case in our problem. The papers that are most closely related to our second-stage problem address the procurement of end-of-use products from the install base. Guide and Van Wassenhove (2001) consider a firm that sets a buyback price to match the supply of cores with the demand for remanufactured components. Bakal and Akcali (2006) also consider a buyback price to match supply with demand. Galbreth and Blackburn (2006) study the interaction between procurement lot size and the firm's sorting policies. Zikopoulos and Tagaras (2007) consider the problem of ordering used products from multiple supply sources with correlated recovery yield. Each of these papers considers a single-stage decision environment. Our work differs from this literature by including a first-stage decision on the final order quantity. In addition, this literature focuses on returns to support the remanufactured product demand, not warranty claims. As a consequence, the models do not capture the negative correlation between return volume and future demand.

In summary, our stage-one problem is similar to final purchase quantity problems in the literature. A key difference is the consideration of a trade-in program that leads to a two-stage decision problem. The final purchase quantity literature has not considered the design of trade-in programs as a mechanism for acquiring used components. Our stage-two problem is similar to the problem of designing a trade-in program that is considered in the marketing and remanufacturing literature. As in this literature, we need to model how features of the trade-in program and other factors influence return volume. However, as noted above, a key difference is that we need to capture how return volumes influence future warranty claims.

MODELS AND ANALYSES

A firm has received a CPOA for a component from a sole-source supplier and must determine the final order quantity q_1 that will be received at time t = 0. The purchase cost per unit is c_1, the inventory holding cost rate is h, and the warranty claim service cost per unit is c_w (e.g., disassembly, component replacement, reassembly, test, and shipping). The difference between the firm's discount rate and the rate of inflation in operating costs and margin is r. The last warranty expires at time t = 1 (i.e., the unit of time is selected so as to normalize the warranty liability horizon to one period).

The component demand rate at time t (due to warranty claims) is $d(t)$, the cumulative demand through period t is $D(t)$, i.e.,

$$D(t) = \int_0^t d(x)\,dx, \tag{1}$$

and the remaining warranty demand is $\bar{D}(t)$, i.e.,

$$\bar{D}(t) = D(1) - D(t). \tag{2}$$

The component demand rate is net of any passive returns of product containing a working component. We assume deterministic demand and focus on identifying the drivers of performance in this setting.

We let $T_1(q_1)$ denote the time that component inventory from the final order reaches zero, or the end of the warranty horizon, whichever is smaller, i.e.,

$$T_1(q_1) = \min\left\{\min\left\{t \mid D(t) \geq q_1\right\}, 1\right\}$$

The total cost to service warranty claims is

$$C_1(q_1) = c_1 q_1 + \int_0^{T_1(q_1)} e^{-rt}\left[h\left(q_1 - D(t)\right) + c_w d(t)\right]dt + C_2\left(T_1(q_1)\right) \tag{3}$$

where $C_2(t)$ is the cost of satisfying warranty claims over time interval [t, 1] given that the final order quantity runs out at time t. The first term in (3) is the component purchase cost of the final order of q_1 units. The second term in (3) is the inventory holding and warranty claim servicing cost, of which the parenthetical term in the integrand is the inventory at time t that is assessed a holding cost rate h.

Second-Stage Trade-in Policy

A firm offering a trade-in program specifies the discount off the purchase price of a new model if the customer returns the old model. The trade-in credit is offered only to customers with product under warranty. Conceivably a firm could offer the trade-in discount to a customer with a product that is no longer under warranty. While such a customer might be willing to trade-in for a lower discount, the tactic of offering a trade-in discount for product not under warranty has two drawbacks. First, there is a risk that the component in the returned product will be faulty. This risk is low for product under warranty because, if it was faulty, the firm would have likely already received a claim. Second, the return of a product under warranty reduces the firm's warranty liability associated with the obsolete component (i.e., the product containing the obsolete component is traded in for a new model of the product).

The firm offers a time-sensitive trade-in to some fraction of warranty holders in each period so as to match the rate of supply with the rate of demand. We refer to this policy as a *matching trade-in policy*. A matching trade-in policy is viable in settings where the firm has access to customer-specific warranty data (i.e., customer contact information for products under warranty). Firms that sell directly to customers are likely to have this level of detail in warranty data.

Before analyzing the trade-in policy, we describe how we model relationships between the trade-in discount, trade-in volume, and trade-in cost. We begin with two assumptions that allow us to define the fraction of customers who accept a trade-in offer as a function of the trade-in discount:

Assumption 1 (A1). *A customer receiving a trade-in offer receives a single take-it-or-leave-it offer and accepts the offer if consumer surplus is positive.*

Assumption 2 (A2). *The valuation of the new model in exchange for the old model under warranty, denoted V, is independent of time and is uniformly distributed and ordered by age of ownership with range normalized to* [0, 1].

An alternative to A1 is to allow multiple trade-in offers to the same customer over time. However, this promotes strategic behavior that greatly complicates the analysis and may work against the interest of the firm (e.g., customer holds out for a better offer). Uniformly distributed valuation (A2) is common in the literature (e.g., Mussa and Rosen 1978, Purohit and Staelin 1994) and results in return volume that is linear in price. A2 also specifies that a consumer who recently purchased the product will have a higher valuation than a consumer who has owned the product for a longer period of time, i.e., customers with older product will accept a lower trade-in offer than customers with newer product.

A firm offering a trade-in program must select the trade-in discount and the rate at which customers are exposed to the trade-in offer (i.e., the trade-in offer rate), both of which may vary with time. The trade-in discount is $c_t(t)$ and the trade-in offer rate is $\nu(t)$ (e.g., $\nu(t)$ is the number of customers receiving a trade-in offer in period t). The contribution margin of a new model of the product is m and the variable cost is c_n, i.e., the new model selling price is $p_n = c_n + m$. Thus, the trade-in price is $c_n + m - c_t(t)$ and, by A1 and A2, the fraction of customers who accept the trade-in offer from among those who receive it is

$$\beta(t) = P\left[V > c_n + m - c_t(t)\right] = 1 - c_n - m + c_t(t) = c_t(t) - (p_n - 1) \tag{4}$$

Rewriting (4) in terms of the trade-in credit,

$$c_t(t) = p_n - \left(1 - \beta(t)\right). \tag{5}$$

We see that the trade-in price is the complement of the acceptance rate $\beta(t)$, i.e.,

$$p_n - c_t(t) = 1 - \beta(t).$$

Note that the new model selling price should be more than the maximum valuation of a trade-in exchange, i.e.,

$$p_n = c_n + m > 1. \tag{6}$$

Condition (6) reflects the practical reality that customers are unlikely to trade in a product under warranty unless there is a trade-in discount. For example, $p_n < 1$ would imply that fraction $1 - p_n$ of customers would be willing to return their product (that is under warranty and functional) and pay full price for the new model.

The value of $p_n - 1$ is a measure of *trade-in resistance*. This value is the minimum trade-in discount that is required before any customers will be willing to return their unit. The larger the value of $p_n - 1$, the greater the market resistance to a trade-in offer, and therefore, the firm is pressured to increase its trade-in offer with a higher value of $c_t(t)$.

In (4), we see that the difference between the trade-in credit, $c_t(t)$, and the trade-in resistance, $p_n - 1$, gives the fraction of those receiving the trade-in offer who accept the offer. Thus, the product return rate $s(t)$ is

$$s(t) = \beta(t)\nu(t) = \left[c_t(t) - (p_n - 1)\right]\nu(t).$$

In general, the specification of trade-in acquisition cost can be challenging due to the effect of cannibalization. We model this effect through parameter γ. The interpretation of γ is relatively straightforward when the difference between the firm's discount rate and the rate of inflation (in costs and margin) is zero (i.e., $r = 0$): γ is the fraction of trade-in customers who would have purchased the new model at full price in the future if the trade-in program was not offered, or repeat purchase rate. If $r > 0$, then the value of the full margin in the future is lower due to the time-value-of-money. All time-value-of-money effects and, more generally, all cannibalization effects are incorporated into the value of parameter γ. Indeed, it is possible for γ to be negative in some settings, e.g., by reducing secondary market supply and thus cannibalization of new product sales. Accordingly, the cost of a component obtained through a trade-in is the reduction in margin through a trade-in sale, which is

$$c_2(t) = c_t(t) - (1-\gamma)m = \beta(t) - \tau \tag{7}$$

where

$\tau = (1 - \gamma)m - (p_n - 1)$.

We refer to the value of τ as the *trade-in potential*, which can be interpreted as the difference between the gain from locking-in disloyal customers via the trade-in offer, $(1 - \gamma)m$, and the market resistance to a trade-in offer, $p_n - 1$. More generally, τ is the marginal profit on trade-in volume at the origin. For example, if $\tau > 0$, then trade-in potential is positive and trade-ins are profitable up to acceptance rate $\beta(t)$. On the other hand, if $\tau < 0$, then trade-in potential is negative and trade-ins are costly from the get-go.

Without loss of generality, we define the product unit such that the warranty population at time zero is 1. In the absence of a trade-in program, the rate at which warranties expire at time t is given by $n(t)$, which is known with certainty (e.g., obtained from company records). While $n(t)$ can conceivably take any functional form, in the interest of parsimony, we limit consideration to the following form that depends on a single parameter, $n \in [0, 1]$:

$$n(t) = \begin{cases} n, & t < 1 \\ 1-n, & t = 1 \end{cases},$$

i.e., warranties expire at rate n over time interval [0, 1) and $1 - n$ warranties expire at time $t = 1$. Figure 1 illustrates three alternative warranty population functions over time in the absence of a trade-in program.

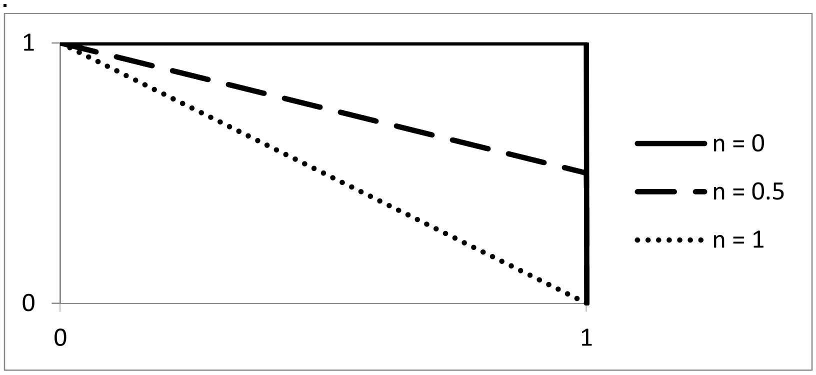

Figure 1. Warranty population over time at different warranty expiration rates n = 0, 0.5, 1.

The case of $n = 1$ reflects a setting where monthly sales of the product is relatively flat near the end of its life-cycle (e.g., warranty expires x months after purchase). The cases of $n < 1$ reflect settings where monthly sales of the product is relatively flat near the end of its life-cycle except for a jump in sales at the end through clearance pricing. The smaller the value of n, the larger the clearance sale volume relative to volume prior to clearance discounting.

With the warranty population characterized, the remaining contributor to demand for components to service warranty claims is the component failure rate. We assume that the failure rate function is constant at value α. This assumption is common in the literature (e.g., Murthy and Rodin 1990, Zhou et al. 2009).

Assumption 3 (A3). *The component failure rate is constant.*

Due to A3, in the absence of a trade-in program, the demand rate is $d(t) = \alpha(1 - nt)$ and demand functions (1) and (2) become

$$D(t) = \alpha\int_0^t (1-nx)\,dx = \alpha t\left[1-0.5nt\right] \tag{8}$$
$$\bar{D}(t) = D(1) - D(t) = \alpha(1-t)\left[1-0.5n(1+t)\right].$$

In both expressions, the term in brackets reflects the degree to which cumulative demand and remaining demand are reduced when the warranty expiration rate n is greater than 0.

We note that a returned unit may contain some value beyond the component that has been phased out. This value can be incorporated into our model as an additional parameter that does not change the structure of the model or the results. In the interest of parsimony, we do not introduce a separate parameter; its value, if significant, is included in parameter m (e.g., if margin is m' and savings generated from other components in a returned product is s, then $m = m' + s/(1 - \gamma)$).

Matching Trade-In Policy

The firm sets the trade-in credit $c_t(t)$ and the trade-in offer rate $\nu(t)$ so that component supply matches component demand over the remainder of the warranty horizon, i.e.,

$$s(t) = \beta(t)\nu(t) = d(t) \tag{9}$$

where $\beta(t) = c_t(t) - \tau$ is the trade-in acceptance rate among those customers exposed to the trade-in offer at time t (see (4)), or the trade-in fraction. Note that $\beta(t)$ must be a valid fraction, i.e.,

$$\beta(t) \in [0, 1], \tag{10}$$

and that a customer receives a trade-in offer no more than once (see A1), i.e.,

$$\int_0^1 \nu(t)\,dt \le 1. \tag{11}$$

The firm's choice of customers who will receive the trade-in offer over time is influenced by A2. Recall that A2 implies that customers with a soon-to-expire warranty are more likely to accept a trade-in offer than customers with a more distant warranty expiration date. In recognition of A2, the firm sends the trade-in offer to customers in order of warranty expiration date.

From (7), the component acquisition cost rate for the matching trade-in policy is

$$c_2^1(t) = \beta(t) + c_n + \gamma m - 1 = \beta(t) - \tau \; . \tag{12}$$

Thus, the cost of the matching trade-in policy is

$$C_2^1(\beta,\nu) = \int_0^1 e^{-rt}\left(\beta(t) - \tau + c_w\right) d(t)\, dt$$

where functions $\beta(t)$ and $\nu(t)$ satisfy (9) – (11). We wish to find the function $\beta(t)$ that minimizes the second-stage cost subject to the relevant constraints. The problem is

$$C_2^1 = \min_{\beta(t)\in[0,1]} \left\{ \int_0^1 e^{-rt}\left(\beta(t) - \tau + c_w\right) d(t)\, dt : \beta(t)\nu(t) = d(t), \int_0^1 \nu(t)\, dt \le 1 \right\}.$$

The following proposition characterizes the optimal solution to the preceding problem.

Proposition 1. *If*

$$e^{\alpha+0.5r} - \left(1 + \frac{r}{2\alpha}\right) e^{\alpha} \le 1 , \tag{13}$$

then the optimal trade-in fraction is

$$\beta(t) = \frac{\alpha\left(1 - e^{-(\alpha+0.5r)}\right) e^{0.5rt}}{\alpha + 0.5r} , \tag{14}$$

the optimal trade-in offer rate is

$$\nu(t) = \left(\frac{(\alpha + 0.5r) e^{-0.5rt}}{1 - e^{-(\alpha+0.5r)}} \right) e^{-\alpha t} ,$$

the demand (and supply rate) is

$$d(t) = \beta(t)\nu(t) = \alpha e^{-\alpha t} ,$$

the total number of units traded in is

$$q_2^1 = 1 - e^{-\alpha} ,$$

and the optimal second-stage cost is

$$C_2^1 = \left(\frac{\alpha\left(1 - e^{-(\alpha+0.5r)}\right)}{\alpha + 0.5r} \right) \left[\left(\frac{\alpha\left(1 - e^{-(\alpha+0.5r)}\right)}{\alpha + r} \right) - \tau + c_w \right].$$

Table 1 shows the maximum value of the failure rate α that satisfies condition (13) for various values of the net discount rate *r*. Recall that α is the failure rate over the duration of the second stage. For example, if the second-stage duration is five years and the annual net discount rate is 5%, then condition (13) holds for a failure rate up to 50% per year (i.e., divide the figures in the row with *r* = 25% by 5). In the computer industry that motivates this work, component failure rates tend to be low (e.g., less than 1%) and the warranty duration is on the order of three to five years. In these settings, the condition given in (13) is highly likely to hold.

r	maximum value of α satisfying (13)
0%	∞
1%	549%
10%	331%
25%	249%
50%	188%
100%	131%
200%	79%

Table 1. Upper limit on component failure rate α for different net discount rates r.

The following corollary gives the optimal solution for the special case of r = 0.

Corollary 1. *If* r = 0, *then the optimal trade-in fraction is*

$$\beta(t) = 1 - e^{-\alpha},$$

the optimal trade-in offer rate is

$$\nu(t) = \left(\frac{\alpha}{1-e^{-\alpha}}\right) e^{-\alpha t}, \tag{15}$$

the demand (and supply rate) is

$$d(t) = \beta(t)\nu(t) = \alpha e^{-\alpha t},$$

the total number of units traded in is

$$q_2^1 = 1 - e^{-\alpha},$$

and the optimal cost is

$$C_2^1 = \left(1-e^{-\alpha}\right)\left(1-e^{-\alpha} - \tau + c_w\right). \tag{16}$$

We see that the optimal solution has a simpler structure when r = 0. In particular, the optimal trade-in fraction $\beta(t)$ is independent of time. This means that the optimal trade-in discount stays constant over the warranty horizon, i.e.,

$$c_t(t) = p_n - e^{-\alpha}, \tag{17}$$

(obtained by substituting $\beta(t)$ into (5)). The term in the parentheses in the right-hand side of (15) is the fraction of the population exposed to the trade-in offer (i.e., $e^{-\alpha t}$ is the warranty population over time), which also stays constant over the warranty horizon.

In contrast, at $r > 0$, we see that the optimal trade-in fraction $\beta(t)$ is increasing in time (see (14)), and consequently, the trade-in discount is increasing in time (e.g., the firm offers higher discounts later in the horizon, which are less costly for the firm due to the positive discount rate). Similarly, the fraction of the population exposed to the trade-in offer is decreasing over time.

First-Stage Problem

Under a policy where the final order quantity is set to match total demand, the total cost is

$$C_1^0 = c_1 q_1^0 + \int_0^1 e^{-rt}\left[h\left(q_1^0 - \alpha\left(t - 0.5nt^2\right)\right) + c_w \alpha\left(1 - nt\right)\right]dt \tag{18}$$

and the final order quantity is

$$q_1^0 = D\left(1\right) = \alpha\left(1 - 0.5n\right)$$

(see (3) and (8)). We refer to this policy, which is identified by superscript 0, as the benchmark policy. The optimal total cost under the matching trade-in program is

$$C_1^1 = c_1 q_1^1 + \int_0^{T_1\left(q_1^1\right)} e^{-rt}\left[h\left(q_1^1 - D\left(t\right)\right) + c_w d\left(t\right)\right]dt + C_2^1\left(T_1\left(q_1^1\right)\right)$$

where

$$q_1^1 = \arg\min_{q_1 \geq 0}\left\{c_1 q_1 + \int_0^{T_1\left(q_1\right)} e^{-rt}\left[h\left(q_1 - D\left(t\right)\right) + c_w d\left(t\right)\right]dt + C_2^1\left(T_1\left(q_1\right)\right)\right\}.$$

Clearly, $C_1^1 \leq C_1^0$ with equality if and only if the optimal second-stage trade-in quantity is zero (i.e., $C_1^1 = C_1^0 \Leftrightarrow q_2^1 = 0$). The following proposition identifies a simple indicator of when it is profitable to supplement the final order quantity with a trade-in program.

Proposition 2. *If* $c_1 > -e^{-r}\tau$ *, then* $C_1^1 < C_1^0$.

A trade-in program clearly adds value when trade-in potential (τ) is nonnegative (because $c_1 > 0$). In the event that trade-in potential is negative, we can be assured that trade-in programs save money if the present value of the magnitude of the trade-in potential (discounted from the end of the warranty horizon) is less than the purchase cost per unit.

Figure 2 illustrates how the final order quantity under the matching trade-in policy differs from the benchmark final order quantity. Figure 3 illustrates the percent savings due to the matching trade-in policy relative to the benchmark. Note that in figures 2 and 3, a matching trade-in program is not used when n = 1 and τ = -0.2 (i.e., $q_1^1 = q_1^0$).

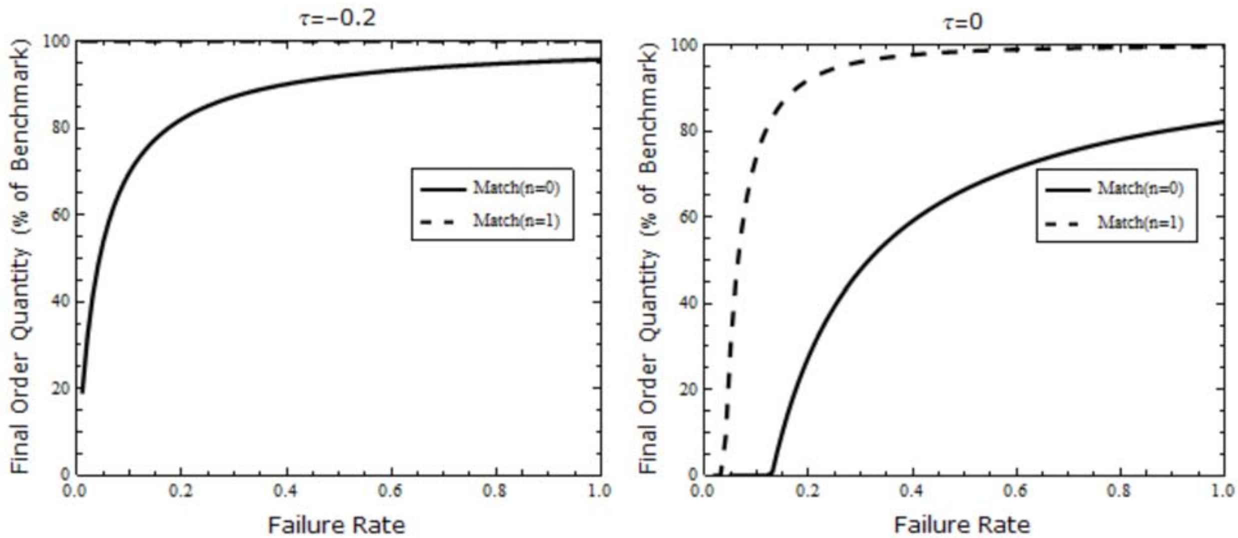

Figure 2. Optimal final order quantities under the matching trade-in policy as a percent of the benchmark final order quantity. The trade-in potential is $\tau = -0.2$ in the left plot and $\tau = 0$ in the right plot. The other parameter values, which are common to both plots are $c_1 = 0.2$, $c_w = 0.1$, $h = 0.07$, $r = 0$.

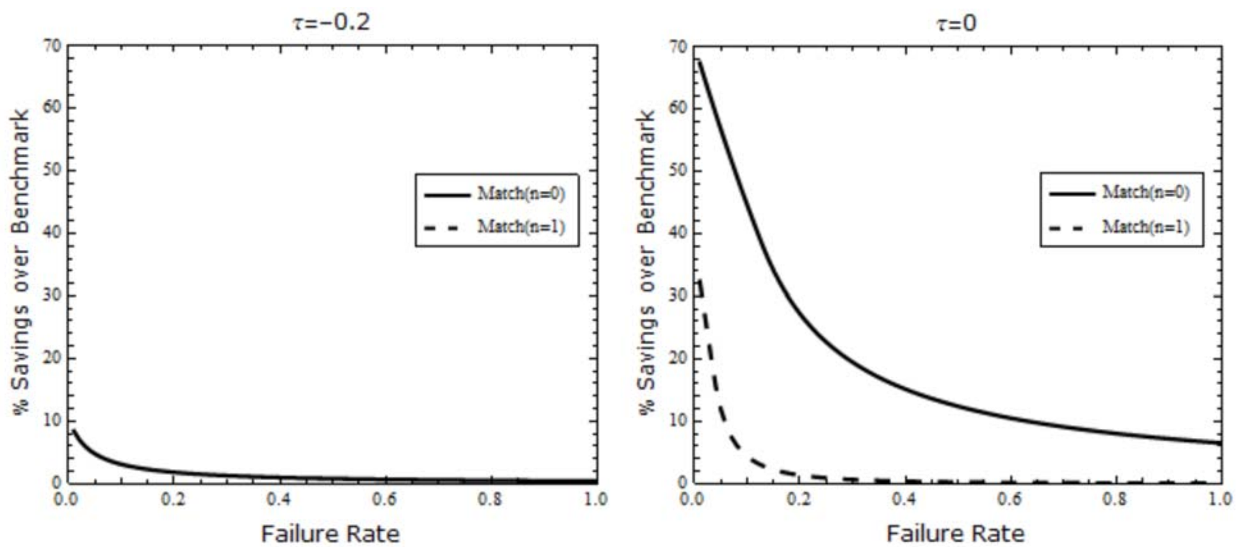

Figure 3. Percent savings in total cost when the benchmark policy is replaced with the matching trade-in policy. The trade-in potential is $\tau = -0.2$ in the left plot and $\tau = 0$ in the right plot. The other parameter values, which are common to both plots are $c_1 = 0.2$, $c_w = 0.1$, $h = 0.07$, $r = 0$.

The left plots of figures 2 and 3 reflect the setting where $c_1 = -\tau$. In this setting, the trade-in acquisition cost with the trade-in credit is so low that the cost of a single returned unit is the same as the cost from the vendor. This setting is extremely unfavorable to a trade-in program and may rarely arise in practice. Nevertheless, even in this unfavorable setting, the trade-in policy is less expensive than the benchmark when $n = 0$. Figure 2 shows that the trade-in policy exploits the flexibility of dividing the source of components between the final order quantity and those that come from trade-ins. The fraction acquired from the vendor via the final order quantity increases as the failure rate increases. This is because the trade-in cost is sensitive to total volume acquired, so a higher fraction of total warranty demand is shifted to the final order quantity.

Figure 3 also illustrates that the percentage savings due to the trade-in policy is significant when the aggregate failure rate is small, and diminishes as the failure rate increases (i.e., as the optimal final order quantity covers an increasing fraction of total demand). However, while percentage savings is decreasing in the failure rate, the cost of the benchmark policy is increasing in the failure rate. In particular, the total cost of the benchmark policy is proportional to the failure rate (see (18)). Depending on parameter values, the absolute savings may be either increasing or decreasing in the failure rate. For the parameter values in Figure 3, for example, absolute savings is nondecreasing in failure rate except for the matching trade-in policy at $n = \tau = 0$, which is initially increasing, then decreasing.

SUMMARY OF LESSONS FOR MANAGERS

We have considered two policies for a CPOA response. The traditional policy, which we refer to as the benchmark policy, is to place a final order that is large enough to cover component warranty demand over the remainder of the warranty horizon. An alternative is to be less aggressive on the final order quantity and, as component inventory approaches zero, acquire additional components through a trade-in program offered to consumers with product under warranty. A trade-in offer is made to only a fraction of the warranty population in each period, with the fractions adding to 100% by the end of the warranty horizon. The trade-in credit and the number of customers who receive the offer in a period are designed to achieve a return response that is sufficient to cover the warranty demand in the upcoming period. A matching trade-in program is viable for companies with relatively accurate and complete information on the warranty population (e.g., customer contact data).

Our analysis leads to two main lessons for managers. First, trade-in programs dominate the benchmark policy and have potential to significantly lower cost. Trade-in programs can be perceived as a new sourcing option, and therefore, incorporating them into the set of alternatives will not increase cost. What is more pertinent for managers are indicators that a trade-in program will generate significant savings. The single most important indicator is trade-in potential. The value of trade-in potential is the difference between two values: (1) the margin from a trade-in transaction, or new-product margin times the probability of a customer not purchasing from the firm if not for the trade-in offer, and (2) the reduction in new product price required to get at least one customer to participate in the trade-in program. If trade-in potential is positive, then the cost of acquiring components via trade-in is negative (at least at low volumes); the firm earns money from acquiring a component via trade-in rather than spending money buying a component from the vendor. Management should examine the difference between trade-in potential and the cost of buying the component from the vendor. These values should not be difficult to estimate, and a large difference is a strong indicator of high savings from a trade-in program.

A secondary indicator of high savings from a trade-in program is the fraction of the warranty population expected to fail prior to warranty expiration. This indicator draws on a more subtle value proposition than the difference in cost between a trade-in-sourced unit and a vendor-sourced unit: a trade-in program reduces warranty claims. Each unit traded in reduces the warranty population of the obsolete product, which translates into fewer warranty claims. And the higher the failure rate, the greater the reduction in warranty claims.

The overarching lesson from our analysis is that the use of a trade-in program to support warranty claims should be considered by management. It is surprising to see that the use of trade-ins to support warranty claims is not discussed in the industry or academic literature. One possible reason is that, relative to the benchmark policy, a trade-in program is more difficult to implement, i.e., the firm has to design and communicate the program. However, it is also the case that trade-in programs are not unusual in practice. Indeed, there is a wide literature in economics and marketing on this topic, and this may hint at the possibility of an organizational barrier. Trade-in programs are typically designed and administered by the marketing group for marketing reasons (e.g., price discrimination, spur sales when a new generation of a product is introduced, etc.). Reducing the cost of warranty claim processing is often outside the mission of a marketing department. Our work can motivate firms to lower such organizational barriers.

REFERENCES

Bakal, I.S., & Akcali, E. (2006). Effects of random yield in remanufacturing with price-sensitive supply and demand. *Production and Operations Management*, 15, 407-420.

Blischke, W., & Murthy, D. (1992). Product warranty management--I: A taxonomy for warranty policies. *European Journal of Operational Research*, 62(2), 127-148.

Bradley, J. R., & Guerrero, H.H. (2009). Lifetime buy decisions with multiple obsolete parts. *Production and Operations Management*, 18(1), 114-126.

Bruce, N., Desai, P., & Staelin, R. (2006). Enabling the willing: Consumer rebates for durable goods. *Marketing Science*, 25(4), 350-366.

Fleischmann, M., van Nunen, J., & Gräve, B. (2003). Integrating closed-loop supply chains and spare-parts management at IBM. *Interfaces*, 33(6), 44-56.

Fortuin, L. (1980). The all-time requirement of spare parts for service after sales–theoretical analysis and practical results. *International Journal of Operations & Production Management*, 1(1), 59-70.

Galbreth, M.R., & Blackburn, J.D. (2006). Optimal acquisition and sorting policies for remanufacturing. *Production and Operations Management*, 15, 384-392.

Guide, V.D.R. Jr., & Van Wassenhove, L.N. (2001). Business aspects of closed-loop supply chain management. V.D.R. Guide Jr., L.N. Van Wassenhove, eds. *Business Aspects of Closed-loop Supply Chain Management.* Pittsburgh, PA: Carnegie Mellon University Press, 17-41.

Inderfurth, K., & Mukherjee, K. (2008). Decision support for spare parts acquisition in post product life cycle. *Central European Journal of Operations Research*, 16(1), 17-42.

Kennedy, W., Patterson, J.W., & Fredendall, L. (2002). An overview of recent literature on spare parts inventories. *International Journal of Production Economics*, 76(2), 201-215.

Linton, J.D. (2008). Assessing the economic rationality of remanufacturing products. *Journal of Product Innovation Management*, 25(3), 287-302.

Minner, S., & Kleber, R. (2001). Optimal control of production and remanufacturing in a simple recovery model with linear cost functions. *OR Spektrum*, 23, 3-42.

Murthy, D. N. P., & Rodin, E.Y. (1990). A new warranty costing model. *Mathematical and Computer Modelling*, 13(9), 59-69.

Murthy, D., Solem, O., & Roren, T. (2004). Product warranty logistics: Issues and challenges. *European Journal of Operational Research*, 156(1), 110-126.

Mussa, M., & Rosen, S. (1978). Monopoly and product quality. *Journal of Economic Theory*, 18(2), 301-317.

Nidumolu, R., Prahalad, C.K., & Rangaswami, M.R. (2009). Why sustainability is now the key driver of innovation. *Harvard Business Review*, 87(9), 56-64

Purohit, D., & Staelin, R. (1994). Rentals, sales, and buybacks: Managing secondary distribution channels. *Journal of Marketing Research*, 31, 325-338.
Rao, R., Narasimhan, O., & John, G. (2009). Understanding the role of trade-ins in durable goods markets: Theory and evidence. *Marketing Science*, 28(5), 950-967.
Ray, S., Boyaci, T., & Aras, N. (2005). Optimal prices and trade-in rebates for durable, remanufacturable products. *Manufacturing Service Operations Management*, 7(3), 208-228.
Seitz, M.A. (2007). A critical assessment of motives for product recovery: The case of engine remanufacturing. *Journal of Cleaner Production*, 15(11-12), 1147-1157
Spengler, T., & Schröter, M. (2003). Strategic Management of Spare Parts in Closed-Loop Supply Chains - A System Dynamics Approach. *Interfaces*, 33(6) 7-17.
Teunter, R.H., & Fortuin, L. (1998). End-of-life service: A case study. *European Journal of Operational Research*, 107(1), 19-34.
Teunter, R.H., & Fortuin, L. (1999). End-of-life service. *International Journal of Production Economics*, 59(1-3), 487-497.
Teunter, R.L., & Haneveld, W.K. (1998). The final order problem. *European Journal of Operational Research*, 107(1), 35-44.
Zhou, Z., Li, Y., & Tang, K. (2009). Dynamic pricing and warranty policies for products with fixed lifetime. *European Journal of Operational Research*, 196(3) 940-948.
Zikopoulos, C., & Tagaras, G. (2007). Impact of uncertainty in the quality of returns on the profitability of a single-period refurbishing operation. *European Journal of Operational Research*, 182(1), 205-225.

APPENDIX

Matching Trade-in Policy Cost for $q_1 \geq 0$

$$
\begin{aligned}
C_1^1(q_1) = {} & q_1\left(c_1 + h\left(\frac{1-e^{-rt_1}}{r}\right)\right) + \alpha\left(\left(c_w - \frac{h}{r}\right)\left(\frac{1-e^{-rt_1}}{r}\right)\right) + \frac{ht_1\alpha e^{-rt_1}}{r} - \\
& n\alpha\left(\left(c_w - \frac{h}{r}\right)\left(\frac{1-e^{-rt_1}}{r^2} - \frac{e^{-rt_1}}{r}\right) + \frac{ht_1^2 e^{-rt_1}}{2r}\right) + \\
& e^{-rt_1}(1-nt_1)\left[\left(\frac{\alpha\left(1-e^{-(\alpha+0.5r)(1-t_1)}\right)}{\alpha+0.5r}\right)^2 - \left(\frac{\alpha\left(1-e^{-(\alpha+0.5r)(1-t_1)}\right)}{\alpha+r}\right)(\tau - c_w)\right]
\end{aligned}
$$

When $r = 0$, the expression reduces to

$$
\begin{aligned}
C_1^1(q_1) = {} & q_1(c_1 + ht_1) + \alpha(c_w - 0.5ht_1)t_1 - 0.5n\alpha(c_w - 0.33ht_1)t_1^2 + \\
& (1-nt_1)\left(1-e^{-\alpha(1-t_1)}\right)\left(\left(1-e^{-\alpha(1-t_1)}\right) + c_w - \tau\right).
\end{aligned}
$$

Proof of Proposition 1. For this proof, we will not use the normalization of $t_1 = 0$ in order to clarify the expressions under a general second-stage starting time t_1, expressions that will appear in our analysis of the first-stage problem. We initially develop the results for the special case of $n = 0$. We will then show that the optimal solution for this special case remains valid when $n > 0$.

Assume n = 0. Let $N(t)$ denote the warranty population at time t. Due to n = 0, we have $N(t) = 1$ for all $t < t_1$. The demand and supply rate over time interval $[t_1, 1]$ is $d(t) = s(t) = \alpha N(t)$ and the warranty population function is

$$N(t) = 1-\int_{t_1}^{t} s(x)\,dx = 1-\int_{t_1}^{t} d(x)\,dx = 1-\int_{t_1}^{t} \alpha N(x)\,dx,\ t\in[t_1,1].$$

We obtain an explicit expression for $N(t)$ by taking the limit of a discrete-time model as the time interval goes to zero. Given time interval $\Delta > 0$, the failure rate per time interval Δ is $\alpha\Delta$, and we have

$$\begin{aligned}
&d(t_1+\Delta) = s(t_1+\Delta) = \alpha\Delta N(t_1+\Delta) = \alpha\Delta\left[1-s(t_1+\Delta)\right]\\
&s(t_1+\Delta) = \frac{\alpha\Delta}{1+\alpha\Delta}\\
&N(t_1+\Delta) = 1-s(t_1+\Delta) = 1-\frac{\alpha\Delta}{1+\alpha\Delta} = \frac{1}{1+\alpha\Delta}\\
&s(t_1+2\Delta) = \alpha\Delta N(t_1+2\Delta) = \frac{\alpha\Delta N(t_1+\Delta)}{1+\alpha\Delta} = \frac{\alpha\Delta}{(1+\alpha\Delta)^2}\\
&N(t_1+2\Delta) = N(t_1+\Delta)-s(t_1+2\Delta) = \frac{1}{1+\alpha\Delta}-\frac{\alpha\Delta}{(1+\alpha\Delta)^2} = \frac{1}{(1+\alpha\Delta)^2}
\end{aligned}$$

and, in general, for integer i

$$N(t_1+i\Delta) = \frac{1}{(1+\alpha\Delta)^i}.$$

Let $t_\Delta = t - t_1 = i\Delta$. Taking the limit as Δ approaches zero,

$$N(t) = \lim_{\Delta\to 0}(1+\alpha\Delta)^{-t_\Delta/\Delta} = e^{-\lim_{\Delta\to 0}\{(t_\Delta/\Delta)\ln(1+\alpha\Delta)\}} = e^{-\lim_{\Delta\to 0}(\alpha t_\Delta/(1+\alpha\Delta))} = e^{-\alpha(t-t_1)} \text{ for } t \in [t_1, 1].$$

Thus, $d(t) = \alpha N(t) = \alpha e^{-\alpha(t-t_1)}$ for $t \in [t_1, 1]$. By substituting

$$\beta(t) = d(t)/\nu(t) = \alpha e^{-\alpha(t-t_1)}/\nu(t) \tag{19}$$

into (5), we see that $c_t(t)$ is decreasing in $\nu(t)$. Therefore, we replace the inequality constraint (11) with equality, and the constraint can be written as

$$\int_{t_1}^{1}\nu(t)\,dt = \int_{t_1}^{1}\frac{\alpha e^{-\alpha(t-t_1)}}{\beta(t)}\,dt = 1, \tag{20}$$

and the second-stage problem can be written as

$$C_2^1 = e^{-rt_1} \min_{\beta(t)\in[0,1]} \left\{ \int_{t_1}^{1} \alpha e^{-(\alpha+r)(t-t_1)} \left(\beta(t) - \tau + c_w\right) dt : \int_{t_1}^{1} \frac{\alpha e^{-\alpha(t-t_1)}}{\beta(t)} dt = 1 \right\}.$$

We solve the following equivalent problem

$$\min_{\beta(t)\in[0,1]} \left\{ \int_{t_1}^{1} e^{-(\alpha+r)(t-t_1)} \beta(t)\, dt : \int_{t_1}^{1} \frac{e^{-\alpha(t-t_1)}}{\beta(t)} dt = \frac{1}{\alpha} \right\}, \tag{21}$$

but we initially relax the bound constraint $\beta(t) \in [0, 1]$ (i.e., unrestricted problem), i.e., we solve

$$\min_{\beta(t)} \left\{ \int_{t_1}^{1} e^{-(\alpha+r)(t-t_1)} \beta(t)\, dt : \int_{t_1}^{1} \frac{e^{-\alpha(t-t_1)}}{\beta(t)} dt = \frac{1}{\alpha} \right\}. \tag{22}$$

After solving the unrestricted problem (22), we identify conditions on parameter values that ensure the solution is also optimal for the restricted problem (21).

To simplify notation, we temporarily let $t_1 = 0$ (we account for the impact of $t_1 > 0$ later). We define

$$y(t) = \int_{0}^{t} \frac{e^{-\alpha x}}{\beta(x)} dx,$$

which implies

$$y'(t) = \frac{e^{-\alpha t}}{\beta(t)}$$

$$\beta(t) = e^{-\alpha t} y'(t)^{-1}. \tag{23}$$

Thus, problem (22) is

$$\min_{y(t)} \left\{ \int_{0}^{1} e^{-(2\alpha+r)t} y'(t)^{-1}\, dt : y(0) = 0, y(1) = \frac{1}{\alpha} \right\}, \tag{24}$$

which can be solved using calculus of variations methods. Let $y^*(t)$ denote the optimal function. We express $y(t)$ in terms of parameter a, $y^*(t)$, and difference function $h(t)$, i.e.,

$$y(t) = y^*(t) + ah(t)$$

and thus $y'(t) = y^{*\prime}(t) + ah'(t)$. For any feasible $y(t)$, we must have $h(0) = h(1) = 0$ (i.e., in order to satisfy the boundary conditions, which are clearly satisfied by the function $y^*(t)$). Let

$$g(a) = \int_{0}^{1} e^{-(2\alpha+r)t} y'(t)^{-1}\, dt = \int_{0}^{1} e^{-(2\alpha+r)t} \left[y^{*\prime}(t) + ah'(t) \right]^{-1} dt.$$

Note that

$$g'(a) = -\int_0^1 e^{-(2\alpha+r)t}\left[y^{*}{}'(t)+ah'(t)\right]^{-2} h'(t)\,dt\,. \tag{25}$$

Applying integration by parts and recognizing $uv\big|_0^1$ = 0 due to $h(0) = h(1) = 0$, we have

$$g'(a) = -\int_0^1 e^{-(2\alpha+r)t}\left[\begin{array}{l}(2\alpha+r)\left(y^{*}{}'(t)+ah'(t)\right)^{-2}+2\\ \left(y^{*}{}'(t)+ah'(t)\right)^{-3}\left(y^{*}{}''(t)+ah''(t)\right)\end{array}\right]h(t)\,dt\,.$$

Since $y^*(t)$ is optimal, we can conclude that for any $h(t)$ (with $h(0) = h(1) = 0$), we must have $g'(0)$ = 0. This implies that the integrand of the above (with a = 0) must be equal to zero at all values of t, i.e., we must have

$$\begin{aligned}
&e^{-(2\alpha+r)t}\left[(2\alpha+r)\left(y^{*}{}'(t)\right)^{-2}+2\left(y^{*}{}'(t)\right)^{-3}y^{*}{}''(t)\right]h(t)=0\\
&\Leftrightarrow\ (2\alpha+r)\,y^{*}{}'(t)+2y^{*}{}''(t)=0\\
&\Leftrightarrow\ y^{*}{}''(t)=-\left(\frac{2\alpha+r}{2}\right)y^{*}{}'(t)
\end{aligned}$$

Solving the differential equation, we get

$$y^{*}{}'(t)=Ae^{-(\alpha+0.5r)t}\,,$$

where A is obtained from the boundary condition, i.e.,

$$y^{*}(1)=\int_0^1 y^{*}{}'(t)\,dt=\frac{1}{\alpha}\Leftrightarrow A=\frac{\alpha+0.5r}{\alpha\left(1-e^{-(\alpha+0.5r)}\right)}$$

Thus

$$y^{*}{}'(t)=\frac{(\alpha+0.5r)e^{-(\alpha+0.5r)t}}{\alpha\left(1-e^{-(\alpha+0.5r)}\right)}\,. \tag{26}$$

The function $y^{*}{}'(t)$ is a unique extremal (i.e., no other function yields $g'(0) = 0$). Taking the derivative of (25) and evaluating at a = 0, we get

$$g''(0) = \int_0^1 2e^{-(2\alpha+r)t}\left[y^{*}{}'(t)\right]^{-3}\left[h'(t)\right]^{2}dt\,.$$

Since $y^{*}{}'(t) > 0$ for all $t \in [0, 1]$, it follows that $g''(0) > 0$, and thus $y^{*}{}'(t)$ solves (24).

Substituting (26) into (23) yields optimal trade-in fraction

$$\beta(t)=e^{-\alpha t}y'(t)^{-1}=\frac{e^{-\alpha t}\alpha\left(1-e^{-(\alpha+0.5r)}\right)e^{(\alpha+0.5r)t}}{\alpha+0.5r}=\frac{\alpha\left(1-e^{-(\alpha+0.5r)}\right)e^{0.5rt}}{\alpha+0.5r},$$

and accounting for $t_1 > 0$ yields optimal trade-in fraction for the unrestricted problem

$$\beta(t)=\frac{\alpha\left(1-e^{-(\alpha+0.5r)(1-t_1)}\right)e^{0.5r(t-t_1)}}{\alpha+0.5r}. \tag{27}$$

Recall that the difference between the unrestricted problem and the restricted problem is that the restricted problem includes the constraint $\beta(t) \in [0, 1]$. From (27) we see that $\beta(t)$ is increasing in t and $\beta(t_1) \in [0, 1]$. Thus, if $\beta(1) \le 1$, then the optimal solution to the unrestricted problem is also optimal for the restricted problem. Note that

$$\beta(1)=\frac{\alpha\left(1-e^{-(\alpha+0.5r)(1-t_1)}\right)e^{0.5r(1-t_1)}}{\alpha+0.5r}\le 1 \Leftrightarrow e^{(\alpha+0.5r)(1-t_1)}-\left(1+\frac{r}{2\alpha}\right)e^{\alpha(1-t_1)}\le 1. \tag{28}$$

Thus, $\beta(t) \in [0, 1]$ if and only if (28) holds. The optimal trade-in offer rate is obtained by substituting (27) into (19) and solving for $\nu(t)$, the trade-in quantity is obtained from $q_2^1=\int_{t_1}^{1}\beta(t)\nu(t)\,dt$, and the optimal cost is obtained by substituting the optimal acceptance rate and trade-in offer rate functions into the cost function:

$$\nu(t)=\left(\frac{(\alpha+0.5r)e^{-0.5r(t-t_1)}}{1-e^{-(\alpha+0.5r)(1-t_1)}}\right)e^{-\alpha(t-t_1)} \tag{29}$$

$$q_2^1=1-e^{-\alpha(1-t_1)}$$

$$C_2^1=e^{-rt_1}\left[\left(\frac{\alpha\left(1-e^{-(\alpha+0.5r)(1-t_1)}\right)}{\alpha+0.5r}\right)^2-\left(\frac{\alpha\left(1-e^{-(\alpha+0.5r)(1-t_1)}\right)}{\alpha+r}\right)(\tau-c_w)\right].$$

In the preceding, we derived the optimal solution under the assumption that $n = 0$. We next show that the solution is also optimal when $n > 0$. Note that warranty population at the beginning of the second stage when the trade-in program goes into effect is $1 - nt_1$. Adapting the solution in (29) to account for the fact that a total $1 - nt_1$ are made during the second stage, we get

$$\nu(t)=(1-nt_1)\left(\frac{(\alpha+0.5r)e^{-0.5r(t-t_1)}}{1-e^{-(\alpha+0.5r)(1-t_1)}}\right)e^{-\alpha(t-t_1)}, \tag{30}$$

i.e., due to our normalization of the population size to 1, (29) gives the optimal fraction of the warranty population that receives the trade-in offer over time.

If $n > 0$, then it is conceivable that some warranties will expire during the second-stage prior to a customer receiving a trade-in offer. If such a scenario cannot occur under the optimal trade-in offer rate given in (30), then the preceding analysis continues to apply. Indeed, as we show below, this is the case.

According to solution (30), the total number of trade-in offers during interval [t_1, t] is

$$f(t) = \int_{t_1}^{t} \nu(x)\,dx = \int_{t_1}^{t} \left(\frac{(\alpha + 0.5r)e^{-0.5r(x-t_1)}}{1 - e^{-(\alpha+0.5r)(1-t_1)}} \right) e^{-\alpha(x-t_1)} dx .$$

Observe that $f(t)$ is a concave increasing function over the interval [t_1, 1] with $f(t_1) = 0$ and $f(1) = 1 - nt_1$. If there was no trade-in policy, the total number of warranties that would expire during interval [t_1, t] is

$$g(t) = \begin{cases} n(t-t_1), & t \in [t_1, 1) \\ 1 - nt_1, & t = 1 \end{cases}.$$

Thus, $f(t) \geq g(t)$ for all $t \in [t_1, 1]$, i.e., no warranties expire during the second stage prior to receipt of a trade-in offer. Therefore, the structure of the optimal solution for the case of $n = 0$ holds for the case of $n > 0$, through the expressions for $\nu(t)$, q_2^1, and C_2^1 are generalized to account for the lower warranty population at the start of the second stage:

$$\beta(t) = \frac{\alpha\left(1 - e^{-(\alpha+0.5r)(1-t_1)}\right)e^{0.5r(t-t_1)}}{\alpha + 0.5r}. \quad (31)$$

$$\nu(t) = (1 - nt_1)\left(\frac{(\alpha + 0.5r)e^{-0.5r(t-t_1)}}{1 - e^{-(\alpha+0.5r)(1-t_1)}} \right) e^{-\alpha(t-t_1)}, \quad (32)$$

$$q_2^1 = (1 - nt_1)\left(1 - e^{-\alpha(1-t_1)}\right) \quad (33)$$

$$C_2^1 = e^{-rt_1}(1 - nt_1)\left[\left(\frac{\alpha\left(1 - e^{-(\alpha+0.5r)(1-t_1)}\right)}{\alpha + 0.5r} \right)^2 - \left(\frac{\alpha\left(1 - e^{-(\alpha+0.5r)(1-t_1)}\right)}{\alpha + r} \right)(\tau - c_w) \right]. \ \square \quad (34)$$

Proof of Proposition 2. The unit acquisition cost under a trade-in program with acceptance rate β is $c_2 = \beta - \tau$ (see (7)), and the acquisition cost at the origin ($\beta = 0$) is $c_2 = -\tau$. Compared to the benchmark, trade-in programs result in lower inventory and fewer total warranty claims. Thus, a necessary condition for $q_2^1 = 0$, is $c_1 \leq -e^{-r}\tau$ (i.e., the firm cannot reduce acquisition cost by acquiring product at the end of the warranty horizon via a trade-in), and the contrapositive of

$$q_2^1 = 0 \Rightarrow c_1 \leq -e^{-r}\tau$$

is

$$c_1 > -e^{-r}\tau \Rightarrow q_2^1 > 0 . \ \square$$