Final Purchase and Trade-in Decisions in Response to a Component Phase-Out Announcement: A Deterministic Analysis

Dwayne Cole
College of Business Administration
Florida A&M University
Tallahassee, FL 32301
dwayne.cole@famu.edu

Burak Kazaz
Martin J. Whitman School of Management
Syracuse University
Syracuse, NY 13244
bkazaz@syr.edu

Scott Webster*
W.P. Carey School of Business
Arizona State University
Tempe, AZ 85287
scott.webster@asu.edu

May 2015

*Corresponding author

We investigate a problem faced by a durable-goods manufacturer of a product that is no longer manufactured but still under warranty. A supplier announces that a component of the product will be phased out and specifies a deadline for the final order. A common response in traditional practice is to place a final order sufficient to cover future warranty claims. We analyze and compare this policy with two policies that use trade-in programs to supplement the final order quantity: (i) A full trade-in policy where the firm issues a one-time offer to the entire population that has the product under warranty, and (ii) a matching trade-in policy where the firm issues a trade-in offer to a fraction of the warranty population in each period.

Our analysis of a deterministic model leads to two main conclusions. First, we find that the savings from the use of a trade-in program can be significant, and we identify easy-to-estimate measures that drive the magnitude of savings. Second, we find that a full trade-in policy is likely to be preferred over a matching trade-in policy. The policy is also easier and more practical to implement. However, if uncertainty in warranty demand is introduced, then a firm may benefit by combining elements of both policies – an initial offer to a sizable fraction of the warranty population followed by periodic offers to remaining segments over time.

Keywords: inventory management; reverse logistics

---

1. Introduction

We investigate a problem faced by a durable-goods manufacturer of a product that is no longer manufactured but still under warranty. A supplier announces that a component of the product will be phased out and specifies a deadline for the final order. The manufacturer projects the component needs for the product under warranty and considers a two-stage decision problem: (i) the size of the final order and, in the event that the final order is less than actual requirements, (ii) the design of a trade-in program for component harvesting.

The importance and prevalence of this problem have increased over time due to shrinking product life-cycles and growth in outsourcing. These trends are especially pronounced in the computer industry where the high pace of change and technical challenges favor supply chains of independent firms with specialized expertis. Indeed, our motivation for this study comes from our discussions with management at a computer manufacturer. The following is a description of the problem by a manager at the firm.

> *What we are doing today for warranty parts is we place an end-of-life buy. The supplier will come to us and say okay, in the next three months we are going to stop producing this part forever, how many do you want? Now typically we would have three to five more years' worth of warranty life that we have to cover for that part when the buyer comes to tell us that. So we then run it through a series of parts planning tools that tells us the demand we will have for that part over the remaining service life, and we assign a service level to that. But naturally considering that it is warranty coverage and we know we can't get the part again, we have to be pretty conservative of how we place that buy. So, by definition, we over purchase on that end-of-life buy. A thought is that if we had something in place such that in those situations where demand for warranty parts ends up greater than we thought, if in those situations we can go out to the install base and proactively identify those units that we would like to have back. We could offer the current customer a very good deal on an upgrade and get those systems back and then tear them down.*

We consider the setting where the component phase-out announcement (CPOA) occurs after the manufacturer has discontinued manufacturing and sales of the parent product. The phased-out component contributes significantly to the value of the product and is not easily or inexpensively obtained from alternative suppliers (e.g., highly engineered and expensive component). These features elevate the importance of managerial attention on an effective response to the CPOA. We investigate how a firm's optimal final order quantity and trade-in program decisions are influenced by industry and market characteristics.

Our main contribution lies in three observations that come from our analysis. First, a trade-in program has potential to significantly reduce a firm's warranty liability. To the best of our knowledge, the use of trade-in programs to support warranty claims has not been considered in the literature. Second, there are two key indicators that savings from a trade-in program will be significant. One indicator is the difference between the component cost and the marginal cost of the first unit acquired via trade-in. A second indicator is the expected fraction of products under warranty that will fail. Both of these values should not be difficult for a firm to estimate. Third, we consider two types of trade-in programs and offer guidance on when a particular program is likely to be preferred.

2. Elements of the Component Phase-Out Announcement Problem

As a new generation of a component is introduced and the volume of the previous generation declines, a supplier eventually ceases to supply the older generation component and announces a time-line for phase out. While it is possible that a CPOA may occur when the manufacturer is still producing a product with the component, we limit consideration to the case where the product is no longer being manufactured. Thus, the final component purchase decision is driven by warranty obligation considerations. Durable-goods manufacturers commonly offer a limited-time warranty to consumers. Dell, for example, provides a replacement warranty on their PCs. Each computer is sold with Dell's Basic Service Plan, which includes a minimum of 12 months of warranty coverage. The plan covers all Dell-branded component parts. A consumer can extend the basic service plan for up to four additional years by paying a fee at the time of purchase.

The CPOA problem can be viewed as a two-stage decision problem. The first-stage decision is the number of components in the final order. After the final order is placed, component demand is realized over time. The second-stage decision, if necessary, is the price discount to be offered on a trade-in. The second-stage decision is required if or when component supply approaches zero, in which case the firm announces a trade-in program that is open to customers with product under warranty.

In some settings, the firm may have access to customer-specific warranty data. In these settings, the trade-in offer can be targeted to specific customers based on product age and time remaining under warranty. In settings where these data are not available, a firm can announce a limited-time trade-in offer for product under warranty to the general public. We consider both settings in our analysis.

A firm interested in component harvesting could offer a buyback program instead of a trade-in program. Buyback programs offer money for used product without the requirement that the consumer purchase a new product from the firm. While we use a trade-in program as the setting of our model, as shown in the online appendix, a buyback model has the same structure. We describe how relevant parameters are affected by a buyback program. All of our results continue to hold if the relevant trade-in parameters are replaced by the corresponding buyback parameters.

3. Related Literature

Most consumer durables come with either a pro-rata refund or a free repair/replacement warranty policy (Blischke and Murthy 1992, Yeh and Fang 2015). Murthy et al. (2004) provide a comprehensive review of various issues associated with warranty management. Warranty claims are driven by the warranty population, usage characteristics, product reliability, and warranty terms. Seitz (2007) reports that the use of recovered components to satisfy warranty claims is a common practice in the automobile and home appliance industries. Cisco began using returns to support warranty claims in 2008. The initiative increased the recovered value from returns by nine-fold, from 5% to 45% (Nidumolu et al. 2009).

There are two streams of research related to our two-stage problem of how a manufacturer determines the size of the final order and designs the trade-in program for component harvesting. One stream, which is related to our second-stage problem, considers the relationships between new product prices, trade-in or buyback prices, product return volumes, and new product purchases (e.g., Guide and Van Wassenhove 2001, Ray et al. 2005, Bakal and Akcali 2006). Our work differs from this literature by including a first-stage decision on the final order quantity. In addition, this literature focuses on returns to support the remanufactured product demand, not warranty claims. As a consequence, a key distinctive feature of our problem is that each returned unit reduces the size of the install base thereby reducing the demand of warranty claims in the future. This dynamic is not present in the existing literature on trade-in and buyback programs.

A second stream considers the challenge of covering component demand after a CPOA. Much of the work in this stream addresses the final order quantity problem in isolation, which corresponds to our first-stage problem (e.g., Fortuin 1980, Teunter and Hansveld 1998, Teunter and Fortuin 1999, Bradley and Guerrero 2009, van der Heijden and Iskander 2013). Most of the papers in this stream apply to a setting where the returns do not affect future demand for the component. There are a few papers where return volume affects future demand, which is a characteristic of our problem (e.g., Krikke and van der Laan 2011, Pourakbar et al. 2014). The models in these papers include a linkage between product replacement and future demand of the component due to removal of an obsolete product from the population. However, return volume is exogenous as the firm passively accepts returns.

One paper that treats return volume as endogenous and that links return volume to future demand, as in this paper, is Kleber et al. (2012). They consider an OEM managing inventory of a component subject to a CPOA. The component is used to repair broken machines for a fee under a service contract. Each failure yields a nonfunctioning component that, if repairable, can be repaired for use in a future machine failure. They formulate the problem as a deterministic mixed-integer linear program. They use numerical analysis to assess the value of supplementing a final order quantity with machines acquired from customers that can be harvested for components. The authors suggest that future research should examine structural properties of optimal solutions via optimal control methods. Our paper deals with a different setting where returns are from consumers with product under warranty and in working order. However our problem does share three basic features: (i) a final order decision, (ii) endogenous returns, and (iii) return volume affecting future demand for components. While we also consider a deterministic model, our modeling approach is distinct; we use optimal control (and other analysis) methods to obtain structural properties of optimal solutions pertaining to two strategies for the timing and quantity of returns to supplement the final order quantity. Our structural properties expose key factors that govern the attractiveness and the form of trade-in programs to support warranty claims.

4. Models and Analysis

We begin by presenting a model of the CPOA problem. The model includes a policy-dependent second-stage cost function. Section 4.1 contains optimal decisions and cost functions for two second-stage trade-in policies. Section 4.2 characterizes the optimal first-stage decisions and costs under each of the second-stage policies. Proofs and derivations are located in the online appendix.

A firm has received a CPOA for a component from a sole-source supplier and must determine the final order quantity q_1 that will be received at time $t = 0$. The purchase cost per unit is c_1, the inventory holding cost rate is h, and the warranty claim service cost per unit is c_w (e.g., disassembly, component replacement, reassembly, test, and shipping). The difference between the firm's discount rate and the rate of inflation in operating costs and margin is r. The last warranty expires at time $t = 1$ (i.e., the unit of time is selected so as to normalize the warranty liability horizon to one period).

The component demand rate at time t (due to warranty claims) is $d(t)$, the cumulative demand through period t is $D(t) = \int_0^t d(x)\,dx$. The component demand rate is net of any passive returns of product containing a working component. We assume deterministic demand and focus on identifying the drivers of performance in this setting. In Section 4.3 we discuss the impact of uncertainty on our conclusions.

We let $T_1(q_1)$ denote the time that component inventory from the final order reaches zero, or the end of the warranty horizon, whichever is smaller, that is,

$$T_1(q_1) = \min\left\{\min\left\{t \mid D(t) = q_1\right\}, 1\right\}$$

The total cost to service warranty claims is

$$C_1(q_1) = c_1 q_1 + \int_0^{T_1(q_1)} e^{-rt}\left[h\big(q_1 - D(t)\big) + c_w d(t)\right]dt + C_2\big(T_1(q_1)\big) \tag{1}$$

where $C_2(t)$ is the cost of satisfying warranty claims over time interval $[t, 1]$ given that the final order quantity runs out at time t. The first term in (1) is the component purchase cost of the final order of q_1 units. The second term in (1) is the inventory holding and warranty claim servicing cost, of which the parenthetical term in the integrand is the inventory at time t that is assessed a holding cost rate h.

4.1 Second-Stage Trade-in Policies

We consider two trade-in policies for servicing warranty claims. A firm offering a trade-in program specifies the discount off the purchase price of a new model if the customer returns the old model. For both policies, the trade-in credit is offered only to customers with product under warranty. Conceivably a firm could offer the trade-in discount to a customer with a product that is no longer under warranty. While such a customer might be willing to trade-in for a lower discount, the tactic of offering a trade-in discount for product not under warranty has two drawbacks. First, there is a risk that the component in the returned product will be faulty. This risk is low for product under warranty because, if it was faulty, the firm would have likely already received a claim. Second, the return of a product under warranty reduces the firm's warranty liability associated with the obsolete component (i.e., the product containing the obsolete component is traded in for a new model of the product).

In one policy, which we call a *full trade-in policy*, the firm launches a time-sensitive promotion to the entire warranty population. The firm only provides a trade-in credit for units returned during a valid offer period. For example, HP offers featured, time-sensitive promotions on certain product categories (e.g., earn a trade-in credit for HP Designjet T1300 between August 1 and October 31). The trade-in discount under this policy is set so that total component supply matches total warranty demand.

An alternative to a one-time trade-in offer to the entire population is to offer a time-sensitive trade-in to some fraction of warranty holders in each period so as to match the rate of supply with the rate of demand. We refer to this policy as a *matching trade-in policy*. A matching trade-in policy is only viable in settings where the firm has access to customer-specific warranty data (i.e., customer contact information for products under warranty). Firms that sell directly to customers are likely to have this level of detail in warranty data, whereas this level of detail is less likely for firms that sell through retail outlets (e.g., warranty registrations are made available to consumers who purchase through retail outlets, but not all consumers fill out and return this information).

Before analyzing the two trade-in policies, we describe how we model relationships between the trade-in discount, trade-in volume, and trade-in cost. We begin with two assumptions that allow us to define the fraction of customers who accept a trade-in offer as a function of the trade-in discount:

Assumption 1 (A1). *A customer receiving a trade-in offer receives a single take-it-or-leave-it offer and accepts the offer if consumer surplus is positive.*

Assumption 2 (A2). *The valuation of the new model in exchange for the old model under warranty, denoted V, is independent of time and is uniformly distributed and ordered by age of ownership with range normalized to* [0, 1].

An alternative to A1 is to allow multiple trade-in offers to the same customer over time. However, this promotes strategic behavior that greatly complicates the analysis and may work against the interest of the firm (e.g., customer holds out for a better offer). Uniformly distributed valuation (A2) is common in the literature (e.g., Mussa and Rosen 1978, Purohit and Staelin 1994) and results in return volume that is linear in price. A2 also specifies that a consumer who recently purchased the product will have a higher valuation than a consumer who has owned the product for a longer period of time (i.e., customers with older product will accept a lower trade-in offer than customers with newer product).

A firm offering a trade-in program must select the trade-in discount and the rate at which customers are exposed to the trade-in offer (i.e., the trade-in offer rate), both of which may vary with time. The trade-in discount is $c_t(t)$ and the trade-in offer rate is $\nu(t)$ (e.g., $\nu(t)$ is the number of customers receiving a trade-in offer in period t). The contribution margin of a new model of the product is m and the variable cost is c_n (i.e., the new model selling price is $p_n = c_n + m$). Thus, the trade-in price is $c_n + m - c_t(t)$ and, by A1 and A2, the fraction of customers who accept the trade-in offer from among those who receive it is

$$\beta(t) = P\left[V > c_n + m - c_t(t)\right] = 1 - c_n - m + c_t(t) = c_t(t) - (p_n - 1). \tag{2}$$

Rewriting (2) in terms of the trade-in credit,

$$c_t(t) = p_n - (1 - \beta(t)). \tag{3}$$

We see that the trade-in price is the complement of the acceptance rate $\beta(t)$, that is,

$$p_n - c_t(t) = 1 - \beta(t).$$

Note that the new model selling price should be more than the maximum valuation of a trade-in exchange, that is,

$$p_n = c_n + m > 1. \tag{4}$$

Condition (4) reflects the practical reality that customers are unlikely to trade in a product under warranty unless there is a trade-in discount. For example, $p_n < 1$ would imply that fraction $1 - p_n$ of customers would be willing to return their product (that is under warranty and functional) and pay full price for the new model.

The value of $p_n - 1$ is a measure of *trade-in resistance*. This value is the minimum trade-in discount that is required before any customers will be willing to return their unit. The larger the value of $p_n - 1$, the greater the market resistance to a trade-in offer, and therefore, the firm is pressured to increase its trade-in offer with a higher value of $c_t(t)$.

In (2), we see that the difference between the trade-in credit, $c_t(t)$, and the trade-in resistance, $p_n - 1$, gives the fraction of those receiving the trade-in offer who accept the offer. Thus, the product return rate $s(t)$ is

$$s(t) = \beta(t)\nu(t) = \left[c_t(t) - (p_n - 1)\right]\nu(t).$$

In general, the specification of trade-in acquisition cost can be challenging due to the effect of cannibalization. We model this effect through parameter γ. The interpretation of γ is relatively straightforward when the difference between the firm's discount rate and the rate of inflation (in costs and margin) is zero (i.e., $r = 0$): γ is the fraction of trade-in customers who would have purchased the new model at full price in the future if the trade-in program was not offered, or repeat purchase rate. If $r > 0$, then the value of the full margin in the future is lower due to the time-value-of-money. All time-value-of-money effects and, more generally, all cannibalization effects are incorporated into the value of parameter γ. Indeed, it is possible for γ to be negative in some settings (e.g., by reducing secondary market supply and thus cannibalization of new product sales). Accordingly, the cost of a component obtained through a trade-in is the reduction in margin through a trade-in sale, which is

$$c_2(t) = c_t(t) - (1-\gamma)m = \beta(t) - \tau \tag{5}$$

where

$$\tau = (1 - \gamma)m - (p_n - 1).$$

We refer to the value of τ as the *trade-in potential*, which can be interpreted as the difference between the gain from locking-in disloyal customers via the trade-in offer, $(1 - \gamma)m$, and the market resistance to a trade-in offer, $p_n - 1$. More generally, τ is the marginal profit on trade-in volume at the origin. For example, if $\tau > 0$, then trade-in potential is positive and trade-ins are profitable up to acceptance rate $\beta(t)$. On the other hand, if $\tau < 0$, then trade-in potential is negative and trade-ins are costly from the get-go.

Without loss of generality, we define the product unit such that the warranty population at time zero is 1. In the absence of a trade-in program, the rate at which warranties expire at time t is given by $n(t)$, which is known with certainty (e.g., obtained from company records). While $n(t)$ can conceivably take any functional form, in the interest of parsimony, we limit consideration to the following form that depends on a single parameter, $n \in [0, 1]$:

$$n(t) = \begin{cases} n, & t < 1 \\ 1-n, & t = 1 \end{cases},$$

i.e., warranties expire at rate n over time interval $[0, 1)$ and $1 - n$ warranties expire at time $t = 1$. The case of $n = 1$ reflects a setting where monthly sales of the product is relatively flat near the end of its life-cycle (e.g., warranty expires x months after purchase). The cases of $n < 1$ reflect settings where monthly sales of the product is relatively flat near the end of its life-cycle except for a jump in sales at the end through clearance pricing. The smaller the value of n, the larger the clearance sale volume relative to volume prior to clearance discounting.

With the warranty population characterized, the remaining contributor to demand for components to service warranty claims is the component failure rate. We assume deterministic failures and focus on identifying the drivers of performance in this setting. We assume that the failure rate is constant at value

α. This assumption is common in the literature (e.g., Murthy and Rodin 1990, Zhou et al. 2009).

Assumption 3 (A3). *The component failure rate is constant.*

Due to A3, in the absence of a trade-in program, the demand rate is $d(t) = \alpha(1 - nt)$. The cumulative and remaining demand functions are

$$D(t) = \int_0^t d(x)\,dx = \alpha \int_0^t (1 - nx)\,dx = \alpha t[1 - 0.5nt] \tag{6}$$

$$\bar{D}(t) = D(1) - D(t) = \alpha(1 - t)\big[1 - 0.5n(1 + t)\big].$$

In both expressions, the term in brackets reflects the degree to which cumulative demand and remaining demand are reduced when the warranty expiration rate n is greater than 0.

We note that a returned unit may contain some value beyond the component that has been phased out. This value can be incorporated into our model as an additional parameter that does not change the structure of the model or the results. In the interest of parsimony, we do not introduce a separate parameter; its value, if significant, is included in parameter m (e.g., if margin is m' and savings generated from other components in a returned product is s, then $m = m' + s/(1 - \gamma)$).

We are now ready to present results for the full and matching trade-in policies. To simplify notation, we assume $T_1(q_1) = 0$ in our presentation of second-stage results. The function $T_1(q_1)$, which captures the dependency of the second-stage duration on the value of q_1, will enter into our analysis of the first-stage decisions and costs (see the online appendix for the derivation of second-stage cost expressions that allow for $T_1(q_1) > 0$).

4.1.1 Second-Stage Policy 1: Full Trade-in Policy

The firm offers a trade-in opportunity to the entire warranty population (population size = 1) at time $t = 0$. Let q_2 denote the number of units that are traded in. At time $t = 0$, the warranty population is reduced from 1 to $1 - q_2$. Due to A2 that implies that the oldest products under warranty are returned, no warranties expire until time

$$t = t_x \equiv \min\{q_2/n, 1\},$$

at which point, warranties begin to expire at rate n until the end of the warranty horizon. Consequently, the demand rate is

$$d(t) = \begin{cases} \alpha(1 - q_2), & t \le t_x \\ \alpha\big(1 - q_2 - n(t - t_x)\big), & t \ge t_x \end{cases} \tag{7}$$

and the total demand is

$$D(1) = \alpha\left(1 - q_2 - 0.5n(1 - t_x)^2\right).$$

Setting supply equal to demand and solving for q_2 yields the return quantity under the full trade-in policy:

$$q_2^1 = \begin{cases} \dfrac{\alpha}{1+\alpha}, & n \le \dfrac{\alpha}{1+\alpha} \\ \left[\left(\dfrac{n}{\alpha}\right)^2 + n(2-n)\right]^{1/2} - \dfrac{n}{\alpha}, & n \ge \dfrac{\alpha}{1+\alpha} \end{cases} \tag{8}$$

(we use a superscript throughout to denote the trade-in policy in effect, that is, superscript 1 refers to the full trade-in policy and superscript 2 refers to the matching trade-in policy).

Recall that $\beta(0)$ is the fraction of customers who accept the trade-in offer from the population who receive the offer. For a full trade-in program, this fraction is the same as q_2^1; we obtain the trade-in discount by substituting $q_2^1 = \beta(0)$ into (3):

$$c_t^1 = q_2^1 + p_n - 1. \tag{9}$$

Substituting (9) into (5) yields the trade-in cost per unit:

$$c_2^1 = q_2^1 - (1 - c_n - \gamma m) = q_2^1 - \tau . \tag{10}$$

Thus, the second-stage cost under a full trade-in policy is

$$C_2^1 = c_2^1 q_2^1 + \int_0^1 e^{-rt}\left[h\left(q_2^1 - D(t)\right) + c_w d(t)\right]dt .$$

If $n \le \alpha/(1+\alpha)$, then the second-stage cost can be expressed rather simply:

$$C_2^1 = \left(\frac{\alpha}{1+\alpha}\right)\left[\frac{\alpha}{1+\alpha} - \tau + c_w\left(\frac{1-e^{-r}}{r}\right) + h\left(\frac{r+e^{-r}-1}{r^2}\right)\right],$$

which reduces to

$$C_2^1 = \left(\frac{\alpha}{1+\alpha}\right)\left[\frac{\alpha}{1+\alpha} - \tau + c_w + \frac{h}{2}\right] \tag{11}$$

when $r = 0$. For the case of $n > \alpha/(1+\alpha)$, the cost expressed in terms of the parameters is very complex and is not illuminating.

4.1.2 Second-Stage Policy 2: Matching Trade-in Policy

The firm sets the trade-in credit $c_t(t)$ and the trade-in offer rate $\nu(t)$ so that component supply matches component demand over the remainder of the warranty horizon, that is,

$$s(t) = \beta(t)\nu(t) = d(t) \tag{12}$$

where $\beta(t) = c_t(t) - \tau$ is the trade-in acceptance rate among those customers exposed to the trade-in offer at time t (see (2)), or the trade-in fraction. Note that $\beta(t)$ must be a valid fraction, that is,

$$\beta(t) \in [0, 1], \tag{13}$$

and that a customer receives a trade-in offer no more than once (see A1), that is,

$$\int_0^1 \nu(t)\,dt \le 1. \tag{14}$$

The firm's choice of customers who will receive the trade-in offer over time is influenced by A2. Recall that A2 implies that customers with a soon-to-expire warranty are more likely to accept a trade-in offer than customers with a more distant warranty expiration date. In recognition of A2, the firm sends the trade-in offer to customers in order of warranty expiration date.

From (5), the component acquisition cost rate for the matching trade-in policy is

$$c_2^2(t) = \beta(t) + c_n + \gamma m - 1 = \beta(t) - \tau\ . \tag{15}$$

Thus, the cost of the matching trade-in policy is

$$C_2^2(\beta,\nu) = \int_0^1 e^{-rt}\left(\beta(t) - \tau + c_w\right) d(t)\,dt$$

where functions $\beta(t)$ and $\nu(t)$ satisfy (12) – (14). We wish to find the function $\beta(t)$ that minimizes the second-stage cost subject to the relevant constraints. The problem is

$$C_2^2 = \min_{\beta(t)\in[0,1]} \left\{ \int_0^1 e^{-rt}\left(\beta(t) - \tau + c_w\right) d(t)\,dt : \beta(t)\nu(t) = d(t), \int_0^1 \nu(t)\,dt \le 1 \right\}.$$

The following proposition characterizes the optimal solution to the preceding problem.

Proposition 1. *If*

$$e^{\alpha+0.5r} - \left(1 + \frac{r}{2\alpha}\right) e^{\alpha} \le 1, \tag{16}$$

then the optimal trade-in fraction is

$$\beta(t) = \frac{\alpha\left(1 - e^{-(\alpha+0.5r)}\right) e^{0.5rt}}{\alpha + 0.5r}, \tag{17}$$

the optimal trade-in offer rate is

$$\nu(t) = \left(\frac{(\alpha + 0.5r) e^{-0.5rt}}{1 - e^{-(\alpha+0.5r)}} \right) e^{-\alpha t},$$

the demand (and supply rate) is

$$d(t) = \beta(t)\nu(t) = \alpha e^{-\alpha t},$$

the total number of units traded in is

$$q_2^2 = 1 - e^{-\alpha},$$

and the optimal second-stage cost is

$$C_2^2 = \left(\frac{\alpha\left(1-e^{-(\alpha+0.5r)}\right)}{\alpha+0.5r}\right)\left[\left(\frac{\alpha\left(1-e^{-(\alpha+0.5r)}\right)}{\alpha+r}\right)-\tau+c_w\right].$$

Table 1 shows the maximum value of the failure rate α that satisfies condition (16) for various values of the net discount rate r. Recall that α is the failure rate over the duration of the second stage. For example, if the second-stage duration is five years and the net annual net discount rate is 5%, then condition (16) holds for a failure rate up to 50% per year (i.e., divide the figures in the row with r = 25% by 5). In the computer industry that is motivating this work, component failure rates tend to be low (e.g., less than 1%) and the warranty duration is on the order of three to five years. In these settings, the condition given in (16) is likely to hold.

r	**maximum value of α satisfying (16)**
0%	∞
1%	549%
10%	331%
25%	249%
50%	188%
100%	131%
200%	79%

Table 1: Upper limit on component failure rate α for different net discount rates r.

The following corollary gives the optimal solution for the special case of $r = 0$.

Corollary 1. *If* $r = 0$, *then the optimal trade-in fraction is*

$$\beta(t) = 1-e^{-\alpha},$$

the optimal trade-in offer rate is

$$\nu(t) = \left(\frac{\alpha}{1-e^{-\alpha}}\right)e^{-\alpha t}, \tag{18}$$

the demand (and supply rate) is

$$d(t) = \beta(t)\nu(t) = \alpha e^{-\alpha t},$$

the total number of units traded in is

$$q_2^2 = 1-e^{-\alpha},$$

and the optimal cost is

$$C_2^2 = \left(1-e^{-\alpha}\right)\left(1-e^{-\alpha}-\tau+c_w\right). \tag{19}$$

We see that the optimal solution has a simpler structure when $r = 0$. In particular, the optimal trade-in fraction $\beta(t)$ is independent of time. This means that the optimal trade-in discount stays constant over the warranty horizon, that is,

$$c_t\left(t\right) = p_n - e^{-\alpha}, \tag{20}$$

(obtained by substituting $\beta(t)$ into (3)). The term in the parentheses in the right-hand side of (18) is the fraction of the population exposed to the trade-in offer (i.e., $e^{-\alpha t}$ is the warranty population over time), which also stays constant over the warranty horizon.

In contrast, at $r > 0$, we see that the optimal trade-in fraction $\beta(t)$ is increasing in time (see (17)), and consequently, the trade-in discount is increasing in time (e.g., the firm offers higher discounts later in the horizon, which are less costly for the firm due to the positive discount rate). Similarly, the fraction of the population exposed to the trade-in offer is decreasing over time.

4.1.3 Comparison of Second-Stage Trade-in Policy Performance

We begin by identifying and describing four factors that interact to influence the relative performance of full and matching trade-in policies. These four factors are: (i) the number of warranty claims, (ii) the component inventory level, (iii) the trade-in potential, and (iv) the discount factor.

First, relative to the matching trade-in policy, the full trade-in policy has the advantage of fewer warranty claims, that is,

$$q_2^1 \le \frac{\alpha}{1+\alpha} < 1 - e^{-\alpha} = q_2^2 \tag{21}$$

(follows from $e^{-x} < (1 + x)^{-1}$ for all $x > 0$). For both policies, the total number of trade-in units is equal to the total warranty claims during the second stage. Warranty claims are lower for the full trade-in policy because the warranty population is reduced at a single time instant $t = 0$, and any unit removed from the warranty population eliminates a possible future warranty claim on the obsolete product. In contrast, the matching trade-in policy reduces the warranty population gradually over the entire warranty horizon. The difference, $q_2^2 - q_2^1$, is increasing in α (up to $\alpha \approx 250\%$) and n, and the cost advantage to the full trade-in policy from this difference is increasing in the warranty service cost rate c_w. Figure 1 illustrates the warranty claims under full and matching policies over different values of α and n.

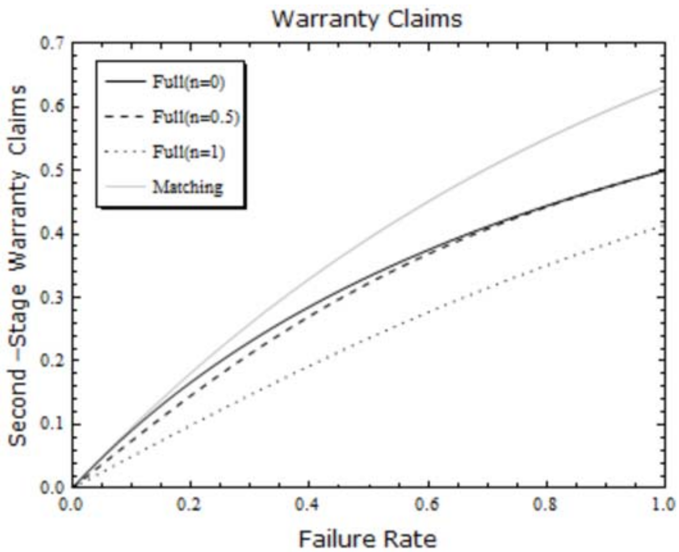

Figure 1: Warranty claims under full and matching trade-in policies for different values of failure rate α and warranty expiration rate *n*.

The second factor that influences the relative performance of full and matching trade-in policies pertains to the component inventory level. The matching trade-in policy has the advantage of no component inventory during the second stage whereas the average second-stage inventory under a full trade-in policy is

$$\bar{I}^1 = \int_0^1 \left(q_2^1 - D(t) \right) dt \,,$$

which for the case of $n \le \alpha/(1+\alpha)$, reduces to

$$\bar{I}^1 = \frac{\alpha}{2(1+\alpha)}$$

(see (7) and (8)). The value of $\bar{I}^1$ is increasing in α and decreasing in *n*, and the cost advantage to the matching trade-in policy from this inventory difference is increasing in the inventory holding cost rate *h*.

The third factor is the trade-in potential, τ, which interacts with trade-in volume to influence which policy results in lower acquisition cost. Recall that the full trade-in policy results in fewer warranty claims, and equivalently, lower trade-in volume. Depending on the value of the trade-in potential, lower trade-in volume can result in lower or higher acquisition cost. For example, when the trade-in potential is sufficiently positive to the point where the firm makes money on each trade-in, then the higher volume of the matching policy can be beneficial. The total acquisition cost under the two policies is

$$\bar{c}_2^1 = c_2^1 q_2^1 = \left(q_2^1 - \tau \right) q_2^1$$

$$\overline{c}_2^2 = \int_0^1 e^{-rt} c_2^2(t)\, d(t)\, dt = \left[\frac{\alpha\left(1-e^{-(\alpha+0.5r)}\right)}{\alpha+0.5r}\right]^2 - \left(\frac{\alpha\tau}{\alpha+r}\right)\left(1-e^{-(\alpha+r)}\right)$$

(see (10), (15), and (17)). For the special case of $r = 0$ and $n \le \alpha/(1+\alpha)$, the total acquisition cost expressions reduce to

$$\overline{c}_2^1 = \left(\frac{\alpha}{1+\alpha}\right)\left(\frac{\alpha}{1+\alpha} - \tau\right)$$

$$\overline{c}_2^2 = \left(1-e^{-\alpha}\right)\left(1-e^{-\alpha}-\tau\right),$$

and we have

$$\overline{c}_2^2 \le \overline{c}_2^1 \text{ if and only if } \tau \ge \frac{\alpha}{1+\alpha} + 1 - e^{-\alpha}$$

(note that $\frac{\alpha}{1+\alpha} + 1 - e^{-\alpha} = q_2^1 + q_2^2 > 0$). Figure 2 illustrates the boundary curves delineating the regions where $\overline{c}_2^2 < \overline{c}_2^1$ and where $\overline{c}_2^2 > \overline{c}_2^1$ on the (α, τ) axis for different values of n and r. The matching trade-in policy tends to be favored when the failure rate is low (α is small) and trade-ins are profitable (trade-in potential τ is large). In this setting, the fact that there are more trade-ins under the matching trade-in policy is an advantage. The right plot of Figure 2 for the case of $n = 0$ shows that the matching trade-in policy results in a lower acquisition costs when both the failure rate and the trade-in potential are small. The reason for this phenomenon relates to the fourth factor that influences the relative performance between the policies.

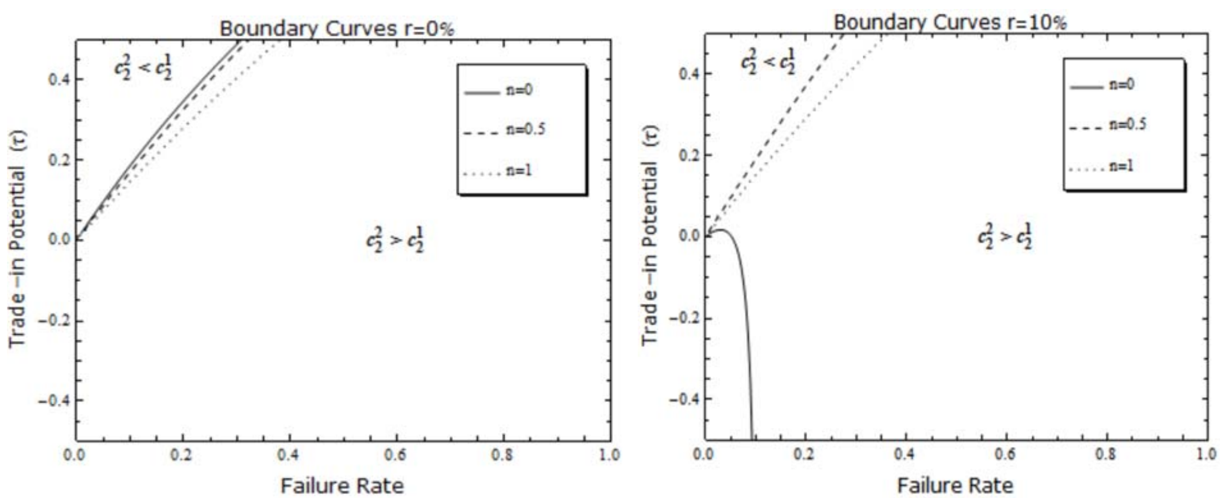

Figure 2: Boundary between lower acquisition cost under the matching trade-in policy ($\overline{c}_2^2 < \overline{c}_2^1$) and lower acquisition cost under the full trade-in policy ($\overline{c}_2^2 > \overline{c}_2^1$) for different values of warranty expiration rate n. The net discount rates in the left and right plots are $r = 0$ and $r = 10\%$, respectively.

The fourth factor is the net discount rate r. The net discount rate plays a subtle role in relative policy performance. Whether a high (or low) value of r favors the full trade-in policy or the matching trade-in policy depends on multiple parameters. Inventory holding costs are incurred continually during the second stage under a full trade-in policy whereas there is no inventory holding cost under a matching trade-in policy. Consequently, an increase in r tilts relative performance in favor of the full trade-in policy through its dampening effect on holding cost. On the other hand, the full trade-in policy incurs the entire acquisition cost at time $t = 0$ and this cost is unaffected by changes in r. Acquisitions are spread out over time under the matching trade-in policy and, while the acquisition rate over the second stage is affected by changes in r, the magnitude of the acquisition cost is reduced due to discounting. Thus, if acquisition costs are positive (negative) under both policies, then an increase in r tilts relative performance in favor of the matching (full) trade-in policy. We see an illustration of this effect in Figure 2. For example, consider the case of $r = 0$ (the left plot in Figure 2). It follows from $\overline{c}_2^1 = q_2^1\left(q_2^1 - \tau\right)$, $\overline{c}_2^2 = q_2^2\left(q_2^2 - \tau\right)$, and $q_2^1 < q_2^2$ that

$$\overline{c}_2^2 < \overline{c}_2^1 \Rightarrow \overline{c}_2^1 < 0,$$

i.e., acquisition cost under both policies is negative when the acquisition cost under the matching trade-in policy is less than the acquisition cost under the full trade-in policy. Now consider the impact of the change from $r = 0$ (left plot) to $r = 10\%$ (right plot). For the case where warranties expire during the warranty horizon (i.e., $n = 0.5$ and $n = 1$), we see that the region of $\overline{c}_2^2 < \overline{c}_2^1$ slightly shrinks (at each value of α) when r increases from 0 to 10%. The case where all warranty expirations occur at the end of the warranty horizon ($n = 0$) presents a different story. The full trade-in policy is more attractive at positive values of τ because all profits from trade-ins are realized at time zero instead of being discounted as realized throughout the warranty horizon. The discounting effect that works against the matching trade-in policy at $\tau > 0$ becomes an advantage when $\tau < 0$. This is illustrated in the region of $\overline{c}_2^2 < \overline{c}_2^1$ that appears in the lower left of the right plot in Figure 2. If the discount rate is high enough for the cases of $n = 0.5$ and $n = 1$, we see similar behavior where the full trade-in policy has a lower acquisition cost for $\tau > 0$ and the matching trade-in policy has a lower acquisition cost at low failure rates for $\tau < 0$ (e.g., for $n = 0.5$, the shift in regions occurs at $r \approx 60\%$, and for $n = 1$, the shift in regions occurs at $r \approx 160\%$).

In order to expose how various factors interact to affect policy preference, consider the case of $r = 0$ and $n \leq \alpha/(1 + \alpha)$. By comparing the expressions for C_2^2 and C_2^1 (see (11) and (19)), it follows that policy dominance is defined by the following:

$$C_2^2 \le C_2^1 \text{ if and only if } h \ge 2\left[\left(\frac{q_2^2}{q_2^1} - 1\right)\left(\frac{q_2^2}{q_2^1} + 1 + c_w - \tau\right)\right]$$

where $\dfrac{q_2^2}{q_2^1} = \dfrac{\left(1-e^{-\alpha}\right)\left(1+\alpha\right)}{\alpha}$.

Figure 3 illustrates the regions of preference for the two policies on the (α, $\tau - c_w$) axis for different values of h when $r = 0$ and $n \le \alpha/(1 + \alpha)$. The figure shows the expanding preferred region of the matching trade-in policy as the inventory holding cost rate increases.

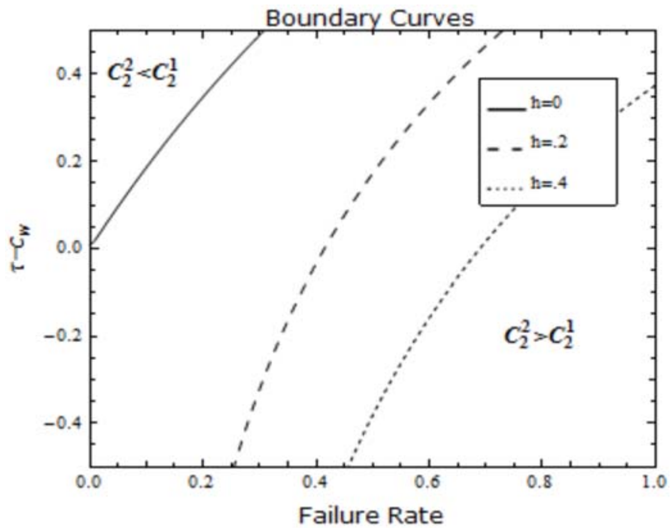

Figure 3: Boundary between lower total cost under the matching trade-in policy ($C_2^2 < C_2^1$) and under the full trade-in policy ($C_2^2 > C_2^1$) for $r = 0$ and $n \le \alpha/(1 + \alpha)$ and different values of inventory holding cost rate h. The vertical axis is 'trade-in potential less warranty service cost' (i.e., $\tau - c_w$).

When $r > 0$ and/or $n > \alpha/(1 + \alpha)$, the inequality $C_2^2 \le C_2^1$ expressed in terms of parameters is complex and is not illuminating. However, the effects of increasing r and n have been exposed in the discussion of figures 1 and 2. To recap, an increase in the warranty expiration rate n has no impact on the matching trade-in policy, whereas an increase in n beyond the threshold $\alpha/(1 + \alpha)$ generally benefits the full trade-in policy (i.e., by reducing the number of warranty claims and trade-ins). If the profit on a trade-in more than covers the warranty processing cost, then it is possible that an increase in n will decrease the attractiveness of the full trade-in (by reducing the profit). An increase in the discount rate benefits the full trade-in policy by reducing the inventory holding cost; the discount rate has no impact on inventory holding cost under the matching policy because there is no inventory. The full trade-in policy incurs all costs associated with acquisition at the beginning of the second-stage, and thus this cost is not affected by increases in the discount rate. The matching trade-in policy defers acquisition costs and thus benefits from increases in the discount rate.

4.2 First-Stage Problem

Under a policy where the final order quantity is set to match total demand, the total cost is

$$C_1^0 = c_1 q_1^0 + \int_0^1 e^{-rt}\left[h\left(q_1^0 - \alpha\left(t - 0.5nt^2\right)\right) + c_w \alpha\left(1 - nt\right)\right]dt \tag{22}$$

and the final order quantity is

$$q_1^0 = D(1) = \alpha(1 - 0.5n)$$

(see (1) and (6)). We refer to this policy, which is identified by superscript 0, as the benchmark policy. The optimal total cost under the full and matching trade-in program is

$$C_1^j = c_1 q_1^j + \int_0^{T_1\left(q_1^j\right)} e^{-rt}\left[h\left(q_1^j - D(t)\right) + c_w d(t)\right]dt + C_2^j\left(T_1\left(q_1^j\right)\right)$$

where

$$q_1^j = \arg\min_{q_1 \ge 0}\left\{c_1 q_1 + \int_0^{T_1(q_1)} e^{-rt}\left[h\left(q_1 - D(t)\right) + c_w d(t)\right]dt + C_2^j\left(T_1\left(q_1\right)\right)\right\} \text{ for } j = 1, 2.$$

Clearly, $C_1^1 \le C_1^0$ and $C_1^2 \le C_1^0$ with equality if and only if the optimal second-stage trade-in quantity is zero (i.e., $C_1^1 = C_1^0 \Leftrightarrow q_2^1 = 0$; $C_1^2 = C_1^0 \Leftrightarrow q_2^2 = 0$). The following proposition identifies a simple indicator of when it is profitable to supplement the final order quantity with a trade-in program.

Proposition 2. *If* $c_1 > -e^{-r}\tau$, *then* $C_1^1 < C_1^0$ *and* $C_1^2 < C_1^0$.

Figure 4 illustrates how the final order quantity under full and matching trade-in policies differs from the benchmark final order quantity. Figure 5 illustrates the percent savings due to full and matching trade-in policies relative to the benchmark. Note that in figures 4 and 5, a matching trade-in program is not used when $n = 1$ and $\tau = -0.2$ (i.e., $q_1^2 = q_1^0$).

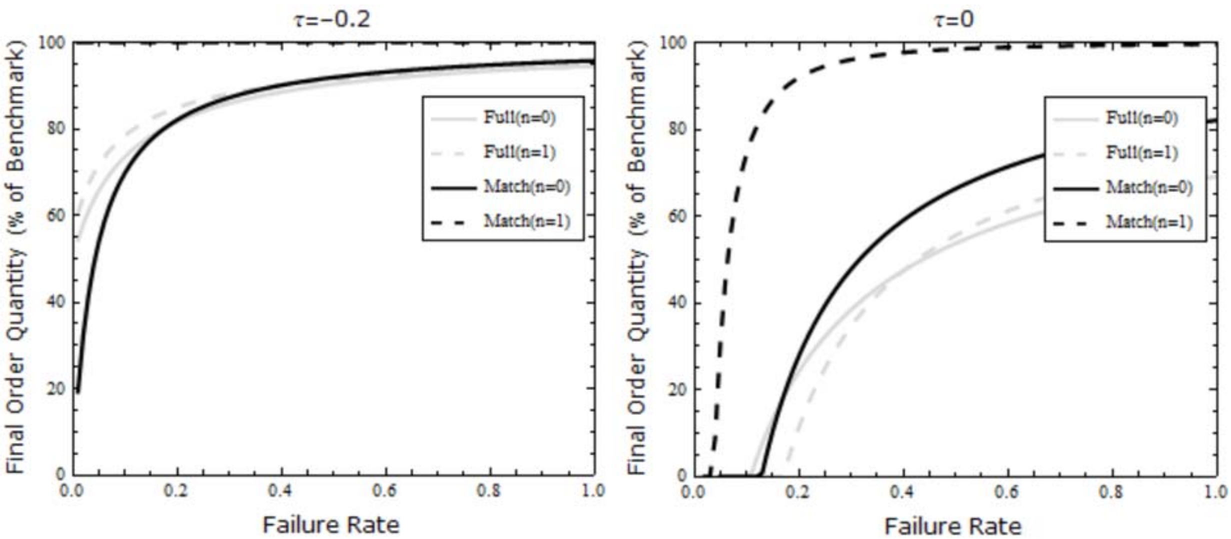

Figure 4: Optimal final order quantities under full and matching trade-in policies as a percent of the benchmark final order quantity. The trade-in potential is $\tau = -0.2$ in the left plot and $\tau = 0$ in the right plot. The other parameter values, which are common to both plots are $c_1 = 0.2$, $c_w = 0.1$, $h = 0.07$, $r = 0$.

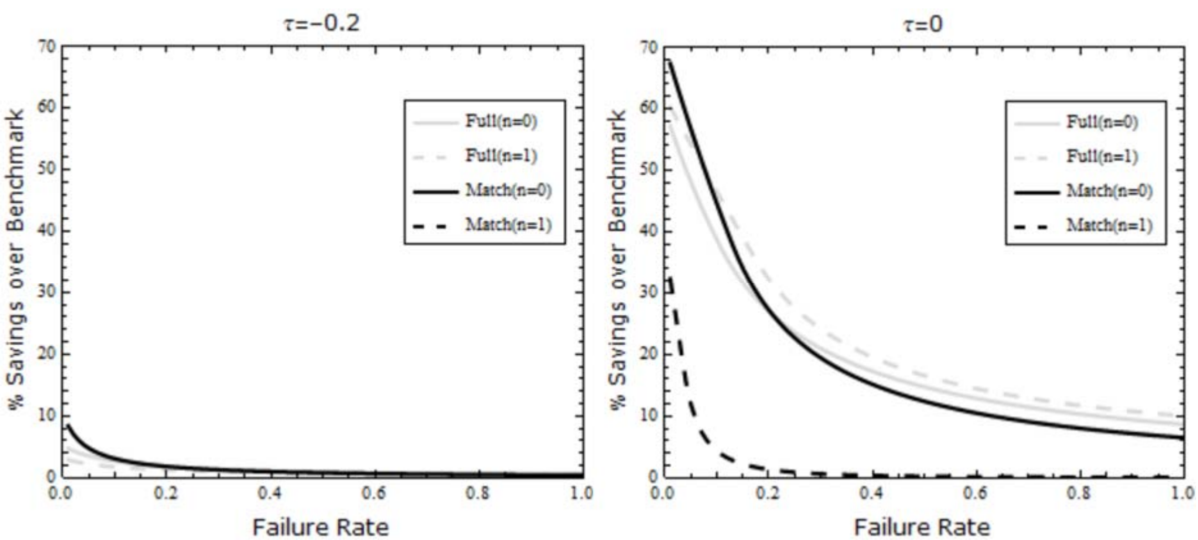

Figure 5: Percent savings in total cost when the benchmark policy is replaced with full and matching trade-in policies. The trade-in potential is $\tau = -0.2$ in the left plot and $\tau = 0$ in the right plot. The other parameter values, which are common to both plots are $c_1 = 0.2$, $c_w = 0.1$, $h = 0.07$, $r = 0$.

The left plots of figures 4 and 5 reflect the setting where $c_1 = -\tau$. In this setting, the trade-in acquisition cost with the trade-in credit is so low that the cost of a single returned unit is the same as the cost from the vendor. This setting is extremely unfavorable to a trade-in program and may rarely arise in practice. Nevertheless, even in this unfavorable setting, trade-in policies are generally less expensive than the benchmark over all failure rates (the exception being the matching policy with $n = 1$). Figure 4 shows that trade-in policies exploit the flexibility of dividing the source of components between the final order quantity and those that come from trade-ins. The fraction acquired from the vendor via the final order quantity increases as the failure rate increases. This is because the trade-in cost is sensitive to total volume acquired, so a higher fraction of total warranty demand is shifted to the final order quantity.

Figure 4 also shows that the final order quantity q_1 is generally larger under the matching trade-in policy than under the full trade-in policy. An exception to this relationship occurs at $n = 0$ and low failure rates. The reason stems from the second-stage inventory carrying cost that arises under a full trade-in policy; there is no second-stage inventory cost under a matching policy (due to zero inventory). The firm compensates for the full trade-in cost disadvantage by placing a larger final order quantity under the full trade-in policy than under the matching trade-in policy. The situation is reversed at larger failure rates. At larger failure rates, the full trade-in policy's advantage of fewer warranty claims more than offsets the inventory carrying cost disadvantage, relative to the matching trade-in policy. In this setting, the firm compensates for the higher second-stage cost under the matching trade-in policy by placing a larger final order quantity than under the full trade-in policy. This relationship holds for all failure rates in Figure 4 when the warranty expiration rate is high (i.e., $n = 1$). This is because an increase in the warranty expiration rate increases the relative cost advantage of the full trade-in policy through a greater reduction in warranty claims (see Figure 1).

Figure 5 exhibits a pattern in cost savings that is similar to Figure 4 for the final order quantity. At $n = 0$ and low failure rates, a matching trade-in policy is less costly because of the elimination of inventory carrying cost. At higher failure rates (and $n = 0$) the situation is reversed because the full trade-in policy's advantage of fewer warranty claims more than offsets the inventory carrying cost disadvantage, relative to the matching trade-in policy. At $n = 1$, however, the full trade-in policy leads to significant percentage savings over the matching policy because an increase in the warranty expiration rate increases the relative cost of advantage of the full trade-in policy through a greater reduction in warranty claims.

Figure 5 also illustrates that the percentage savings due to the trade-in policies is significant when the aggregate failure rate is small, and diminishes as the failure rate increases (i.e., as the optimal final order quantity covers an increasing fraction of total demand). However, while percentage savings is decreasing in the failure rate, the cost of the benchmark policy is increasing in the failure rate. In particular, the total cost of the benchmark policy is proportional to the failure rate (see (22)). Depending on parameter values, the absolute savings may be either increasing or decreasing in the failure rate. For the parameter values in Figure 5, for example, absolute savings is nondecreasing in failure rate except for the matching trade-in policy at $n = \tau = 0$, which is initially increasing, then decreasing.

We conclude this section with an observation on the impact of relaxing a constraint that is present in the optimization problem of the trade-in policies. Both policies set total supply (i.e., initial order quantity plus trade-in volume) equal to demand. When trade-in potential (τ) and the cost of processing a warranty claim (c_w) are very high, the firm may benefit from acquiring more components than what are needed to satisfy demand. For example, a very attractive full trade-in discount may induce nearly the entire population to trade in their product and almost eliminate future warranty claims. Consideration of this effect increases the attractiveness of trade-in programs relative to the benchmark, and due to its more dramatic effect on future demand, tends to favor a full trade-in policy over a matching trade-in policy.

4.3 Impact of Uncertainty

The analysis in the preceding sections is based on a deterministic model. The introduction of demand uncertainty introduces a new cost due to a mismatch between component supply and demand, and this mismatch cost is increasing in uncertainty.

Incorporating uncertainty into the model greatly reduces its tractability. However, the basic cost impact of demand uncertainty on the benchmark policy versus the trade-in policies is clear. The benchmark policy uses the final order to cover demand over the entire warranty horizon and thus suffers the greatest exposure to demand uncertainty. Exposure to demand uncertainty under the trade-in programs is limited to uncertainty in demand over the second stage. Consequently, the introduction of demand uncertainty increases the attractiveness of trade-in programs compared to the benchmark policy.

The negative effects of warranty demand uncertainty on the three policies are influenced by two main costs. Let c_2^0 denote incremental cost over c_w of satisfying a warranty claim in the second stage by means other than a trade-in program. For example, if there is a third-party supplier of the component, then c_2^0 is the purchase cost. Alternatively, the firm may wish replace an old model under warranty with a new model, in which case $c_2^0 + c_w$ is the cost of this transaction. Let c_3 denote the component disposal cost at end of the warranty horizon (e.g., $c_3 < 0$ indicates a positive salvage value for the component). The higher the values of c_2^0 and c_3, the greater the negative effect of demand uncertainty on the three policies.

There are two factors that influence how the relative attractiveness of full and matching trade-in programs shift when demand uncertainty is introduced. The magnitude of demand uncertainty over the second stage is affected by the variation in time-to-failure of an individual component and the size of the warranty population. The full trade-in program has the advantage of a smaller warranty population, leading to lower demand uncertainty over the second stage. A disadvantage is that trade-in terms are set at the beginning of the second stage. A matching trade-in policy, on the other hand, has the advantage of adjusting trade-in terms throughout the second-stage as needed to better align supply with demand. Due to this increased responsiveness, a matching trade-in policy is positioned to more effectively mitigate the mismatch costs than the full trade-in policy.

Another form of uncertainty that arises in practice is uncertainty in trade-in return volume. Return volume uncertainty has a greater effect on the full trade-in policy than the matching trade-in policy. The matching policy takes advantage of observed return volume in response to trade-in offers over time. Indeed, while deterministic analysis tends to favor a full trade-in policy, consideration of uncertainty points to a hybrid approach: A firm offers a trade-in policy to a sizeable fraction of the warranty population to gain the benefit of a meaningful drop in the warranty population, then makes periodic offers to remaining segments over time to gain the benefit of postponement, yielding lower uncertainty in demand and return volume.

5. Summary

5.1 Lessons for Managers

Our analysis leads to two main lessons for managers. First, trade-in programs dominate the benchmark policy and have potential to significantly lower cost. Trade-in programs can be perceived as a new sourcing option, and therefore, incorporating them into the set of alternatives will not increase cost. What is more pertinent for managers are indicators that a trade-in program will generate significant savings. The single most important indicator is trade-in potential. The value of trade-in potential is the difference between two values: (i) the margin from a trade-in transaction, or new-product margin times the probability of a customer not purchasing from the firm if not for the trade-in offer, and (ii) the reduction in new product price required to get at least one customer to participate in the trade-in program. If trade-in potential is positive, then the cost of acquiring components via trade-in is negative (at least at low volumes); the firm earns money from acquiring a component via trade-in rather than spending money buying a component from the vendor. Management should examine the difference between trade-in potential and the cost of buying the component from the vendor. These values should not be difficult to estimate, and a large difference is a strong indicator of high savings from a trade-in program.

A secondary indicator of high savings from a trade-in program is the fraction of the warranty population expected to fail prior to warranty expiration. This indicator draws on a more subtle value proposition than the difference in cost between a trade-in-sourced unit and a vendor-sourced unit: a trade-in program reduces warranty claims. Each unit traded in reduces the warranty population of the obsolete component, which translates into fewer warranty claims. And the higher the failure rate, the greater the reduction in warranty claims.

If a trade-in program is pursued, the next question is what type of trade-in program? Comparative analysis of full and matching trade-in programs suggests a second main lesson for managers—a one-time trade-in discount offered to a sizable portion of the warranty population, possibly followed by offers to remaining segments of the population over the remainder of the warranty horizon, is likely to be the preferred alternative. A large initial offering with consequent returns translates into a meaningful drop in the warranty population, and thus future warranty claims. The initial offering may be followed by periodic smaller offerings to other segments to help mitigate the effects of uncertainty in both warranty claims and return volume. In some settings, a single one-time offering (i.e., a full trade-in policy) may be the only viable alternative. As noted above, the implementation of a matching trade-in program requires relatively detailed information on the warranty population (e.g., at a minimum, data on units under warranty with expiration dates by geographical region) that may not be available to some firms.

5.2 Future Research

There are a number of fruitful directions for future research. First, our models are deterministic. Extending the analysis to accommodate uncertainty is a worthy endeavor. Second, there are questions on how to effectively forecast warranty claims and return volumes. Third, we have focused on uncovering insights for management through analysis of stylized models. A natural extension may focus on the development of decision support models for use by practitioners. Fourth, there is a growing industry of reverse logistics firms. This raises the question of how and when to effectively outsource the management of returns, component harvesting, and warranty claim processing.

6. References

Bakal, I.S., & Akcali, E. (2006). Effects of random yield in remanufacturing with price-sensitive supply and demand. *Production and Operations Management* 15(3) 407-420.

Blischke, W., & Murthy, D. (1992). Product warranty management--I: A taxonomy for warranty policies. *European Journal of Operational Research* 62(2) 127-148.

Bradley, J.R., & Guerrero, H.H. (2009). Lifetime buy decisions with multiple obsolete parts. *Production and Operations Management* 18(1) 114-126.

Fortuin, L. (1980). The all-time requirement of spare parts for service after sales–theoretical analysis and practical results. *International Journal of Operations & Production Management* 1(1) 59-70.

Guide, V.D.R. Jr., & Van Wassenhove, L.N. (2001). Business aspects of closed-loop supply chain management. V.D.R. Guide Jr., L.N. Van Wassenhove, eds. *Business Aspects of Closed-loop Supply Chain Management.* Carnegie Mellon University Press, Pittsburgh, PA.17-41.

Kleber, R., Schulz, T. & Voigt, G. (2012). Dynamic buy-back for product recovery in end-of-life spare parts procurement. *International Journal of Production Research* 50(6) 1476-1488.

Krikke, H., & van der Laan, E. (2011). Last time buy and control policies with phase-out returns: A case study in plant control systems. *International Journal of Production Research* 47(17) 5183-5206.

Murthy, D.N.P., & Rodin, E.Y. (1990). A new warranty costing model. *Mathematical and Computer Modelling* 13(9) 59-69.

Murthy, D., Solem, O., & Roren, T. (2004). Product warranty logistics: Issues and challenges. *European Journal of Operational Research* 156(1) 110-126.

Mussa, M., & Rosen, S. (1978). Monopoly and product quality. *Journal of Economic Theory* 18(2) 301-317.

Nidumolu, R., Prahalad, C.K., & Rangaswami, M.R. (2009). Why sustainability is now the key driver of innovation. *Harvard Business Review* 87(9) 56-64.

Pourakbar, M., van der Laan, E., & Dekker, R. (2014). End-of-life inventory problem with phaseout returns. *Production and Operations Management* 23(9) 1561-1576.

Purohit, D., & Staelin, R. (1994). Rentals, sales, and buybacks: Managing secondary distribution channels. *Journal of Marketing Research* 31 325-338.

Ray, S., Boyaci, T., & Aras, N. (2005). Optimal prices and trade-in rebates for durable, remanufacturable products. *Manufacturing & Service Operations Management* 7(3) 208-228.

Seitz, M.A. (2007). A critical assessment of motives for product recovery: The case of engine remanufacturing. *Journal of Cleaner Production* 15(11-12) 1147-1157.

Teunter, R.L., & Fortuin, L. (1999). End-of-life service. *International Journal of Production Economics* 59(1-3) 487-497.

Teunter, R. L., & Haneveld, W.K. (1998). The final order problem. *European Journal of Operational Research* 107(1) 35-44.

van der Heijden, M., & Iskandar, B.P. (2013). Last time buy decisions for products sold under warranty. *European Journal of Operational Research* 224(2) 302-312.

Yeh, C.-W., & Fang, C.-C. (2015). Optimal decision for warranty with consideration of marketing and production capacity. *International Journal of Production Research*, to appear.

Zhou, Z., Li, Y., & Tang, K. (2009). Dynamic pricing and warranty policies for products with fixed lifetime. *European Journal of Operational Research* 196(3) 940-948.

Online Appendix

A Structurally Equivalent Buyback Model

This sections describes how incorporating buyback programs into our model leads to a structurally equivalent model. Recall that V is the random valuation associated with the transaction of returning the used product in exchange for the new generation of the product, and that the return fraction at trade-in discount c_t is

$$P[V > p_n - c_t] = \beta = c_t - (p_n - 1).$$

The value of V can also be interpreted as the reduction in valuation of a used product relative to the new product, and $p_n - V$ is the residual value of the used product given that it is traded in. The trade-in transaction requires that the customer purchase the new generation product from the firm. Let b denote the increase in V if this requirement is removed, that is, the random valuation of the used product is

$$p_n - (V + b),$$

and a customer receives positive surplus from returning the product and receiving a cash amount c_b if and only if

$$P[c_b > p_n - (V + b)] = P[V > p_n - c_b - b] = \beta = c_b + b - (p_n - 1). \tag{1}$$

Recall also that the cost of a component obtained through a trade-in program for a given return fraction β is

$$c_2 = c_t - (1 - \gamma)m = \beta - (1 - \gamma)m + (p_n - 1).$$

Under a buyback program there is no benefit of locking in disloyal customers (captured by $(1 - \gamma)m$ in a trade-in program), and thus the cost of a component obtained through a buyback program for a given return fraction β is

$$c_b = \beta - b + (p_n - 1)$$

(see (1)). Accordingly, replacing $(1 - \gamma)m$ by b in the following analysis leads to companion results for a buyback program (equivalently, replacing trade-in resistance $\tau = (1 - \gamma)m - (p_n - 1)$ by buyback resistance $\tau_b = b - (p_n - 1)$). Note that a buyback program is more cost effective than a trade-in program if and only if

$$b > (1 - \gamma)m$$

(i.e., the reduction in valuation associated with the forced purchase more than offsets the margin from disloyal customers).

Full Trade-in Policy Cost for $q_1 \geq 0$

In the body of the manuscript we presented an expression for the second-stage cost under a full trade-in policy for the special case of $q_1 = 0$. As noted in Section 4.1, in the absence of a trade-in program, the demand rate is $d(t) = \alpha(1 - nt)$ and the cumulative demand and remaining demand functions are

$$D(t) = \alpha\int_0^t (1-nx)\,dx = \alpha t\left[1-0.5nt\right]$$

$$\bar{D}(t) = D(1) - D(t) = \alpha(1-t)\left[1-0.5n(1+t)\right].$$

In this section, we derive the generalized cost expression that allows for $q_1 \in [0, D(1)]$.

Let $N(t)$ denote the warranty population at time $t \in [0, 1]$. The initial warranty population is normalized to 1 (i.e., $N(0) = 1$). Given that there is no trade-in program, the warranty population at time t is

$$N(t) = 1 - \int_0^t n(x)\,dx = 1 - nt\,.$$

For any given final order quantity $q_1 \in [0, D(1)]$, the run-out time $t_1 = T(q_1)$, by definition, satisfies

$$q_1 = D(t_1) = \alpha t_1(1 - 0.5nt_1).$$

If $n = 0$, then

$$T_1(q_1) = q_1/\alpha.$$

If $n \in (0, 1]$, then solving the quadratic equation for t_1 yields two roots:

$$\frac{1}{n}\left[1 \pm \left(1-2n\frac{q_1}{\alpha}\right)^{1/2}\right].$$

For any $n \in (0, 1]$, the larger of the two roots is greater than 1, which is infeasible, and thus the run-out time function is

$$T_1(q_1) = \frac{1}{n}\left[1 - \left(1-2n\frac{q_1}{\alpha}\right)^{1/2}\right]$$

and the warranty population at the run-out time $t_1 = T_1(q_1)$ is

$$N(T_1(q_1)) = 1 - nT_1(q_1) = \left(1-2n\frac{q_1}{\alpha}\right)^{1/2}.$$

For a given trade-in quantity q_2 at run-out time t_1, no warranties expire until time

$$t_x \equiv \min\{t_1 + q_2/n, 1\},$$

at which point, warranties begin to expire at rate n until the end of the warranty horizon. Consequently, the demand rate is

$$d(t) = \begin{cases} \alpha\left(N(t_1) - q_2\right), & t \in [t_1, t_x] \\ \alpha\left(N(t_1) - q_2 - n(t - t_x)\right), & t \in [t_x, 1] \end{cases}$$

and the demand during the second stage is

$$D(1) - D(t_1) = \int_{t_1}^{1} \alpha\left(N(t_1) - q_2\right)dx - \int_{t_x}^{1} \alpha n\left(t - t_x\right)dx$$

$$= \alpha\left(N(t_1) - q_2\right)\left(1 - t_1\right) - 0.5\alpha n\left(1 - t_x\right)^2$$

If $t_x = 1$, then solving $q_2 = D(1) - D(t_1)$ for q_2 yields

$$q_2^1 = \frac{\alpha(1-t_1)}{1+\alpha(1-t_1)} N(t_1) = \frac{\alpha(1-t_1)}{1+\alpha(1-t_1)}(1 - nt_1).$$

Note that if $n = 0$, then $t_x = 1$. If $t_x < 1$ (and $n > 0$), then the solving $q_2 = D(1) - D(t_1)$ for q_2 yields two roots:

$$\frac{n}{\alpha}\left[-1 \pm \left(1 + \alpha(1-t_1)\left[\alpha(1-t_1) + \frac{2\alpha}{n}(1-n)\right]\right)^{1/2}\right],$$

both of which are real and one of which is negative (and infeasible). Thus, if $t_x < 1$, then

$$q_2^1 = \frac{n}{\alpha}\left[\left(1 + \alpha(1-t_1)\left[\alpha(1-t_1) + \frac{2\alpha}{n}(1-n)\right]\right)^{1/2} - 1\right].$$

Solving $1 = t_1 + q_2^1 / n = t_1 + \frac{1}{\alpha}\left[\left(1 + \alpha(1-t_1)\left[\alpha(1-t_1) + \frac{2\alpha}{n}(1-n)\right]\right)^{1/2} - 1\right]$ for n yields $n = \alpha/(1 + \alpha)$, and thus, $t_x \le 1$ if and only if $n \ge \alpha/(1 + \alpha)$. Combining the above results for different values of t_x, we have

$$q_2^1 = \begin{cases} \dfrac{\alpha(1-t_1)}{1+\alpha(1-t_1)}(1-nt_1), & n \le \dfrac{\alpha}{1+\alpha} \\[2ex] \dfrac{n}{\alpha}\left[\left(1 + \alpha(1-t_1)\left[\alpha(1-t_1) + \dfrac{2\alpha}{n}(1-n)\right]\right)^{1/2} - 1\right], & n \ge \dfrac{\alpha}{1+\alpha} \end{cases}.$$

Substituting

$$t_1 = T_1(q_1) = \begin{cases} \dfrac{q_1}{\alpha}, & n = 0 \\[2ex] \dfrac{1}{n}\left[1 - \left(1 - 2n\dfrac{q_1}{\alpha}\right)^{1/2}\right], & n \in (0,1] \end{cases}$$

into the above yields q_2^1 as a function of q_1, which when substituted into the cost expression yields the cost of a full trade-in policy as a function of the final order quantity q_1:

$$C_1^1(q_1) = c_1 q_1 + \int_0^{T_1(q_1)} e^{-rt}\left[h\left(q_1 - D(t)\right) + c_w d(t)\right]dt + c_2^1 q_2^1 + \int_{T(q_1)}^{1} e^{-rt}\left[h\left(q_2^1 - D(t)\right) + c_w d(t)\right]dt,$$

which after algebra, becomes

$$C_1^1(q_1) = q_1\left(c_1 + h\left(\frac{1-e^{-rt_1}}{r}\right)\right) + \alpha\left(\left(c_w - \frac{h}{r}\right)\left(\frac{1-e^{-rt_1}}{r}\right)\right) + \frac{ht_1\alpha e^{-rt_1}}{r} -$$
$$n\alpha\left(\left(c_w - \frac{h}{r}\right)\left(\frac{1-e^{-rt_1}}{r^2} - \frac{e^{-rt_1}}{r}\right) + \frac{ht_1^2 e^{-rt_1}}{2r}\right) +$$
$$e^{-rt_1}\begin{cases} q_2^1\left(q_2^1 + h\left(\frac{e^{-rt} - e^{-r}}{r}\right) - \tau\right) + \\ \alpha\left(c_w\left(\frac{e^{-rt} - e^{-r}}{r}\right) - h\left(\frac{e^{-rt} - e^{-r}\left(1 - r\left(1 - t_1\right)\right)}{r^2}\right)\right)\left(1 - nt_1 - q_2^1\right) & n \le \frac{\alpha}{1+\alpha} \\ q_2^1\left(q_2^1 + h\left(\frac{e^{-rt} - e^{-r}}{r}\right) - \tau\right) + \\ \alpha\left(c_w\left(\frac{e^{-rt} - e^{-r}}{r}\right) - h\left(\frac{e^{-rt} - e^{-r}\left(1 - r\left(1 - t_1\right)\right)}{r^2}\right)\right)\left(1 - nt_1 - q_2^1\right) - & n \ge \frac{\alpha}{1+\alpha} \\ n\alpha e^{-r}\left(c_w\left(\frac{e^{r(1-t_x)} - 1}{r^2} - \frac{1-t_x}{r}\right) - h\left(\frac{e^{r(1-t_x)} - 1}{r^3} - \frac{1-t_x}{r^2} - \frac{\left(1-t_x\right)^2}{2r}\right)\right) \end{cases}$$

When r =0, the expression reduces to

$$C_1^1(q_1) = q_1\left(c_1 + ht_1\right) + \alpha\left(c_w - 0.5ht_1\right)t_1 - 0.5n\alpha\left(c_w - 0.33ht_1\right)t_1^2 +$$
$$\begin{cases} q_2^1\left(q_2^1 + h\left(1-t_1\right) - \tau\right) + \alpha\left(c_w - 0.5h\left(1-t_1\right)\right)\left(1-t_1\right)\left(1 - nt_1 - q_2^1\right) & n \le \frac{\alpha}{1+\alpha} \\ q_2^1\left(q_2^1 + h\left(1-t_1\right) - \tau\right) + \alpha\left(c_w - 0.5h\left(1-t_1\right)\right)\left(1-t_1\right)\left(1 - nt_1 - q_2^1\right) - & n \ge \frac{\alpha}{1+\alpha} \\ 0.5n\left(c_w - 0.33h\left(1-t_x\right)\right)\left(1-t_x\right)^2 \end{cases}$$

Matching Trade-in Policy Cost for $q_1 \geq 0$

$$C_1^2(q_1) = q_1\left(c_1 + h\left(\frac{1-e^{-rt_1}}{r}\right)\right) + \alpha\left(\left(c_w - \frac{h}{r}\right)\left(\frac{1-e^{-rt_1}}{r}\right)\right) + \frac{ht_1\alpha e^{-rt_1}}{r} -$$
$$n\alpha\left(\left(c_w - \frac{h}{r}\right)\left(\frac{1-e^{-rt_1}}{r^2} - \frac{e^{-rt_1}}{r}\right) + \frac{ht_1^2 e^{-rt_1}}{2r}\right) +$$
$$e^{-rt_1}\left(1 - nt_1\right)\left[\left(\frac{\alpha\left(1 - e^{-(\alpha+0.5r)(1-t_1)}\right)}{\alpha + 0.5r}\right)^2 - \left(\frac{\alpha\left(1 - e^{-(\alpha+0.5r)(1-t_1)}\right)}{\alpha + r}\right)\left(\tau - c_w\right)\right]$$

When r = 0, the expression reduces to

$$C_1^2(q_1) = q_1\left(c_1 + ht_1\right) + \alpha\left(c_w - 0.5ht_1\right)t_1 - 0.5n\alpha\left(c_w - 0.33ht_1\right)t_1^2 +$$
$$\left(1 - nt_1\right)\left(1 - e^{-\alpha(1-t_1)}\right)\left(\left(1 - e^{-\alpha(1-t_1)}\right) + c_w - \tau\right).$$

Proof of Proposition 1. For this proof, we will not use the normalization of $t_1 = 0$ in order to clarify the expressions under a general second-stage starting time t_1, expressions that will appear in our analysis of the first-stage problem. We initially develop the results for the special case of $n = 0$. We will then show that the optimal solution for this special case remains valid when $n > 0$.

Assume $n = 0$. Let $N(t)$ denote the warranty population at time t. Due to $n = 0$, we have $N(t) = 1$ for all $t < t_1$. The demand and supply rate over time interval $[t_1, 1]$ is $d(t) = s(t) = \alpha N(t)$ and the warranty population function is

$$N(t) = 1 - \int_{t_1}^{t} s(x)\,dx = 1 - \int_{t_1}^{t} d(x)\,dx = 1 - \int_{t_1}^{t} \alpha N(x)\,dx,\ t \in [t_1, 1].$$

We obtain an explicit expression for $N(t)$ by taking the limit of a discrete-time model as the time interval goes to zero. Given time interval $\Delta > 0$, the failure rate per time interval Δ is $\alpha\Delta$, and we have

$$d(t_1+\Delta) = s(t_1+\Delta) = \alpha\Delta N(t_1+\Delta) = \alpha\Delta\left[1 - s(t_1+\Delta)\right]$$

$$s(t_1+\Delta) = \frac{\alpha\Delta}{1+\alpha\Delta}$$

$$N(t_1+\Delta) = 1 - s(t_1+\Delta) = 1 - \frac{\alpha\Delta}{1+\alpha\Delta} = \frac{1}{1+\alpha\Delta}$$

$$s(t_1+2\Delta) = \alpha\Delta N(t_1+2\Delta) = \frac{\alpha\Delta N(t_1+\Delta)}{1+\alpha\Delta} = \frac{\alpha\Delta}{(1+\alpha\Delta)^2}$$

$$N(t_1+2\Delta) = N(t_1+\Delta) - s(t_1+2\Delta) = \frac{1}{1+\alpha\Delta} - \frac{\alpha\Delta}{(1+\alpha\Delta)^2} = \frac{1}{(1+\alpha\Delta)^2}$$

and, in general, for integer i

$$N(t_1+i\Delta) = \frac{1}{(1+\alpha\Delta)^i}.$$

Let $t_\Delta = t - t_1 = i\Delta$. Taking the limit as Δ approaches zero,

$$N(t) = \lim_{\Delta\to 0}(1+\alpha\Delta)^{-t_\Delta/\Delta} = e^{-\lim_{\Delta\to 0}\{(t_\Delta/\Delta)\ln(1+\alpha\Delta)\}} = e^{-\lim_{\Delta\to 0}(\alpha t_\Delta/(1+\alpha\Delta))} = e^{-\alpha(t-t_1)} \text{ for } t \in [t_1, 1].$$

Thus, $d(t) = \alpha N(t) = \alpha e^{-\alpha(t-t_1)}$ for $t \in [t_1, 1]$. By substituting

$$\beta(t) = d(t)/\nu(t) = \alpha e^{-\alpha(t-t_1)} / \nu(t) \tag{2}$$

into $c_t(t) = p_n - (1 - \beta(t))$, we see that $c_t(t)$ is decreasing in $\nu(t)$. Therefore, we replace the inequality constraint $\int_0^1 \nu(t)\,dt \le 1$ with equality, and the constraint can be written as

$$\int_{t_1}^{1} \nu(t)\,dt = \int_{t_1}^{1} \frac{\alpha e^{-\alpha(t-t_1)}}{\beta(t)}\,dt = 1, \tag{3}$$

and the second-stage problem can be written as

$$C_2^2 = e^{-rt_1} \min_{\beta(t)\in[0,1]} \left\{ \int_{t_1}^{1} \alpha e^{-(\alpha+r)(t-t_1)} \left(\beta(t) - \tau + c_w\right) dt : \int_{t_1}^{1} \frac{\alpha e^{-\alpha(t-t_1)}}{\beta(t)}\,dt = 1 \right\}.$$

We solve the following equivalent problem

$$\min_{\beta(t)\in[0,1]} \left\{ \int_{t_1}^{1} e^{-(\alpha+r)(t-t_1)} \beta(t)\,dt : \int_{t_1}^{1} \frac{e^{-\alpha(t-t_1)}}{\beta(t)}\,dt = \frac{1}{\alpha} \right\}, \tag{4}$$

but we initially relax the bound constraint $\beta(t) \in [0, 1]$ (i.e., unrestricted problem), that is, we solve

$$\min_{\beta(t)} \left\{ \int_{t_1}^{1} e^{-(\alpha+r)(t-t_1)} \beta(t)\,dt : \int_{t_1}^{1} \frac{e^{-\alpha(t-t_1)}}{\beta(t)}\,dt = \frac{1}{\alpha} \right\}. \tag{5}$$

After solving the unrestricted problem (5), we identify conditions on parameter values that ensure the solution is also optimal for the restricted problem (4).

To simplify notation, we temporarily let $t_1 = 0$ (we account for the impact of $t_1 > 0$ later). We define

$$y(t) = \int_0^t \frac{e^{-\alpha x}}{\beta(x)}\,dx,$$

which implies

$$y'(t) = \frac{e^{-\alpha t}}{\beta(t)}$$

$$\beta(t) = e^{-\alpha t} y'(t)^{-1}. \tag{6}$$

Thus, problem (5) is

$$\min_{y(t)} \left\{ \int_0^1 e^{-(2\alpha+r)t} y'(t)^{-1}\,dt : y(0) = 0, y(1) = \frac{1}{\alpha} \right\}, \tag{7}$$

which can be solved using calculus of variations methods. Let $y^*(t)$ denote the optimal function. We express $y(t)$ in terms of parameter a, $y^*(t)$, and difference function $h(t)$, that is,

$$y(t) = y^*(t) + ah(t)$$

and thus $y'(t) = y^{*\prime}(t) + ah'(t)$. For any feasible $y(t)$, we must have $h(0) = h(1) = 0$ (i.e., in order to satisfy the boundary conditions, which are clearly satisfied by the function $y^*(t)$). Let

$$g(a) = \int_0^1 e^{-(2\alpha+r)t} y'(t)^{-1}\,dt = \int_0^1 e^{-(2\alpha+r)t} \left[y^{*\prime}(t) + ah'(t) \right]^{-1} dt.$$

Note that

$$g'(a) = -\int_0^1 e^{-(2\alpha+r)t}\left[y^{*}{}'(t)+ah'(t)\right]^{-2} h'(t)\,dt\,. \tag{8}$$

Applying integration by parts and recognizing $uv\big|_0^1 = 0$ due to $h(0) = h(1) = 0$, we have

$$g'(a) = -\int_0^1 e^{-(2\alpha+r)t}\begin{bmatrix}(2\alpha+r)\left(y^{*}{}'(t)+ah'(t)\right)^{-2}+2\\ \left(y^{*}{}'(t)+ah'(t)\right)^{-3}\left(y^{*}{}''(t)+ah''(t)\right)\end{bmatrix}h(t)\,dt\,.$$

Since $y^*(t)$ is optimal, we can conclude that for any $h(t)$ (with $h(0) = h(1) = 0$), we must have $g'(0) = 0$. This implies that the integrand of the above (with $a = 0$) must be equal to zero at all values of t, that is, we must have

$$e^{-(2\alpha+r)t}\left[(2\alpha+r)\left(y^{*}{}'(t)\right)^{-2}+2\left(y^{*}{}'(t)\right)^{-3}y^{*}{}''(t)\right]h(t)=0$$

$$\Leftrightarrow\ (2\alpha+r)\,y^{*}{}'(t)+2y^{*}{}''(t)=0$$

$$\Leftrightarrow\ y^{*}{}''(t)=-\left(\frac{2\alpha+r}{2}\right)y^{*}{}'(t)$$

Solving the differential equation, we get

$$y^{*}{}'(t)=Ae^{-(\alpha+0.5r)t},$$

where A is obtained from the boundary condition, i.e.,

$$y^{*}(1)=\int_0^1 y^{*}{}'(t)\,dt=\frac{1}{\alpha}\Leftrightarrow A=\frac{\alpha+0.5r}{\alpha\left(1-e^{-(\alpha+0.5r)}\right)}$$

Thus

$$y^{*}{}'(t)=\frac{(\alpha+0.5r)e^{-(\alpha+0.5r)t}}{\alpha\left(1-e^{-(\alpha+0.5r)}\right)}\,. \tag{9}$$

The function $y^{*}{}'(t)$ is a unique extremal (i.e., no other function yields $g'(0) = 0$). Taking the derivative of (8) and evaluating at $a = 0$, we get

$$g''(0)=\int_0^1 2e^{-(2\alpha+r)t}\left[y^{*}{}'(t)\right]^{-3}\left[h'(t)\right]^{2}dt\,.$$

Since $y^{*\prime}(t) > 0$ for all $t \in [0, 1]$, it follows that $g''(0) > 0$, and thus $y^{*}{}'(t)$ solves (7).

Substituting (9) into (6) yields optimal trade-in fraction

$$\beta(t)=e^{-\alpha t}y'(t)^{-1}=\frac{e^{-\alpha t}\alpha\left(1-e^{-(\alpha+0.5r)}\right)e^{(\alpha+0.5r)t}}{\alpha+0.5r}=\frac{\alpha\left(1-e^{-(\alpha+0.5r)}\right)e^{0.5rt}}{\alpha+0.5r},$$

and accounting for $t_1 > 0$ yields optimal trade-in fraction for the unrestricted problem

$$\beta(t)=\frac{\alpha\left(1-e^{-(\alpha+0.5r)(1-t_1)}\right)e^{0.5r(t-t_1)}}{\alpha+0.5r}. \tag{10}$$

Recall that the difference between the unrestricted problem and the restricted problem is that the restricted problem includes the constraint $\beta(t) \in [0, 1]$. From (10) we see that $\beta(t)$ is increasing in t and $\beta(t_1) \in [0, 1]$. Thus, if $\beta(1) \leq 1$, then the optimal solution to the unrestricted problem is also optimal for the restricted problem. Note that

$$\beta(1)=\frac{\alpha\left(1-e^{-(\alpha+0.5r)(1-t_1)}\right)e^{0.5r(1-t_1)}}{\alpha+0.5r}\leq 1 \Leftrightarrow e^{(\alpha+0.5r)(1-t_1)}-\left(1+\frac{r}{2\alpha}\right)e^{\alpha(1-t_1)}\leq 1. \tag{11}$$

Thus, $\beta(t) \in [0, 1]$ if and only if (11) holds. The optimal trade-in offer rate is obtained by substituting (10) into (2) and solving for $\nu(t)$, the trade-in quantity is obtained from $q_2=\int_{t_1}^{1}\beta(t)\nu(t)\,dt$, and the optimal cost is obtained by substituting the optimal acceptance rate and trade-in offer rate functions into the cost function:

$$\nu(t)=\left(\frac{(\alpha+0.5r)e^{-0.5r(t-t_1)}}{1-e^{-(\alpha+0.5r)(1-t_1)}}\right)e^{-\alpha(t-t_1)} \tag{12}$$

$$q_2=1-e^{-\alpha(1-t_1)}$$

$$C_2^2=e^{-rt_1}\left[\left(\frac{\alpha\left(1-e^{-(\alpha+0.5r)(1-t_1)}\right)}{\alpha+0.5r}\right)^2-\left(\frac{\alpha\left(1-e^{-(\alpha+0.5r)(1-t_1)}\right)}{\alpha+r}\right)(\tau-c_w)\right].$$

In the preceding, we derived the optimal solution under the assumption that $n = 0$. We next show that the solution is also optimal when $n > 0$. Note that warranty population at the beginning of the second stage when the trade-in program goes into effect is $1 - nt_1$. Adapting the solution in (12) to account for the fact that a total $1 - nt_1$ are made during the second stage, we get

$$\nu(t)=(1-nt_1)\left(\frac{(\alpha+0.5r)e^{-0.5r(t-t_1)}}{1-e^{-(\alpha+0.5r)(1-t_1)}}\right)e^{-\alpha(t-t_1)}, \tag{13}$$

i.e., due to our normalization of the population size to 1, (12) gives the optimal fraction of the warranty population that receives the trade-in offer over time.

If $n > 0$, then it is conceivable that some warranties will expire during the second-stage prior to a customer receiving a trade-in offer. If such a scenario cannot occur under the optimal trade-in offer rate given in (13), then the preceding analysis continues to apply. Indeed, as we show below, this is the case.

According to (13), the total number of trade-in offers during interval $[t_1, t]$ is

$$f(t) = \int_{t_1}^{t} \nu(x)\,dx = \int_{t_1}^{t} \left(\frac{(\alpha + 0.5r)e^{-0.5r(x-t_1)}}{1 - e^{-(\alpha+0.5r)(1-t_1)}} \right) e^{-\alpha(x-t_1)} dx\,.$$

Observe that $f(t)$ is a concave increasing function over the interval $[t_1, 1]$ with $f(t_1) = 0$ and $f(1) = 1 - nt_1$. If there was no trade-in policy, the total number of warranties that would expire during interval $[t_1, t]$ is

$$g(t) = \begin{cases} n(t - t_1), & t \in [t_1, 1) \\ 1 - nt_1, & t = 1 \end{cases}.$$

Thus, $f(t) \geq g(t)$ for all $t \in [t_1, 1]$ (i.e., no warranties expire during the second stage prior to receipt of a trade-in offer). Therefore, the structure of the optimal solution for the case of $n = 0$ holds for the case of $n > 0$, through the expressions for $\nu(t)$, q_2, and C_2^2 are generalized to account for the lower warranty population at the start of the second stage:

$$\beta(t) = \frac{\alpha\left(1 - e^{-(\alpha+0.5r)(1-t_1)}\right)e^{0.5r(t-t_1)}}{\alpha + 0.5r}\,. \tag{14}$$

$$\nu(t) = (1 - nt_1)\left(\frac{(\alpha + 0.5r)e^{-0.5r(t-t_1)}}{1 - e^{-(\alpha+0.5r)(1-t_1)}} \right) e^{-\alpha(t-t_1)}, \tag{15}$$

$$q_2 = (1 - nt_1)\left(1 - e^{-\alpha(1-t_1)}\right) \tag{16}$$

$$C_2^2 = e^{-rt_1}(1 - nt_1)\left[\left(\frac{\alpha\left(1 - e^{-(\alpha+0.5r)(1-t_1)}\right)}{\alpha + 0.5r} \right)^2 - \left(\frac{\alpha\left(1 - e^{-(\alpha+0.5r)(1-t_1)}\right)}{\alpha + r} \right)(\tau - c_w) \right]. \ \square \tag{17}$$

Proof of Proposition 2. The unit acquisition cost under a trade-in program with acceptance rate β is $c_2 = \beta - \tau$, and the acquisition cost at the origin ($\beta = 0$) is $c_2 = -\tau$. Compared to the benchmark, trade-in programs result in lower inventory and fewer total warranty claims. Thus, a necessary condition for $q_2^1 = q_2^2 = 0$, is $c_1 \leq -e^{-r}\tau$ (i.e., the firm cannot reduce acquisition cost by acquiring product at the end of the warranty horizon via a trade-in), and the contrapositive of

$$q_2^1 = q_2^2 = 0 \Rightarrow c_1 \leq -e^{-r}\tau$$

is

$$c_1 > -e^{-r}\tau \Rightarrow q_2^1 > 0, q_2^2 > 0\,. \ \square$$