Online Companion

Technical Note – Price-Setting Newsvendor Problems with Uncertain Supply and Risk Aversion

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Appendix

In this appendix, we use the following to denote expected profit for P_S and P_D:

$$\bar{S}(p,q) \equiv E\left[\tilde{S}(p,q,Y)\right]$$

$$\bar{D}(p,q) \equiv E\left[\tilde{D}(p,q,Z)\right].$$

We use subscripts on functions to denote partial derivatives, e.g., $\bar{S}_p = \partial\bar{S} / \partial p$.

A.1 Examples

This section presents four examples. The first three examples are referred to in the manuscript. Example 4 is used in the proof of Proposition 1.

Example 1: $q^o(p) < q^*(p)$ with $[y_L, y_H] \not\subset [(c+s)/(p+s), (c+s)/s]$ for P_S

$c = 40$, $p = 60$, $s = 0$, $d(p) = 100 - p$, $[y_L, y_H] = [0, 2] \not\subset [40/60, \infty)$, $Y \sim$ Uniform[0, 2], $U(x) = 1 - e^{-0.000001x}$: $q^o(p) = 24.4927 < q^*(p) = 24.4949$

Example 2: $p^o(q) > p^*(q)$ with combined multiplicative-additive form of demand for P_D

$c = 1$, $q = 20$, $d(p, z) = (10200 - 100p)z + 101 - p^{1/2}$, $Z \sim$ Uniform[0, 2], $U(x) = 1 - e^{-0.0000000001x}$: $p^o(q) = 94.0430 < p^*(q) = 93.9916$.

Example 3: Instances of (1) $p^o < p^*$, $q^o > q^*$ and (2) $p^o > p^*$, $q^o < q^*$ for P_D and P_S

P_D: (1) $c = 70$, $d(p, z) = (100 - p)z$, $Z \sim$ Uniform[0, 2], $U(x) = 1 - e^{-0.000000005x}$: $p^o = 89.187 < p^* = 89.226$, $q^o = 4.6590 > q^* = 4.6430$. (2) $c = 10$, $d(p, z) = (100 - p)z$, $Z \sim$ Uniform[0.5, 1.5], $U(x) = 1 - e^{-0.0001x}$: $p^o = 56.7841 > p^* = 56.7230$, $q^o = 56.5599 < q^* = 57.2860$.

P_S: (1) $c = 70$, $d(p) = 100 - p$, $Y \sim$ Uniform[0, 2], $U(x) = 1 - e^{-0.0000001x}$: $p^o = 85.3122 < p^* = 85.3132$, $q^o = 8.1080 > q^* = 8.1069$. (2) $c = 40$, $d(p) = 100 - p$, $Y \sim$ Uniform[0,2], $U(x) = 1 - e^{-0.000001x}$: $p^o = 71.3548 > p^* = 71.3475$, $q^o = 19.1286 < q^* = 19.1334$.

Example 4: Concave $\bar{S}(p,q)$ and non-concave $S(p, q)$ for P_S

We define a problem instance where $\bar{S}(p,q)$ is concave and $\varepsilon(p, q) \geq 1/2$ over convex set **X** and where $S(p, q)$ is not concave over **X**, i.e., there are points in **X** where the determinant of the Hessian of $S(p, q)$ is negative. Let

$d(p) = 1 - p$

$\phi_Y(y) = 1/\sigma$, $y \in [1 - 0.5\sigma, 1 + 0.5\sigma]$, $\sigma \in (1, 2]$ (i.e., Y is a uniform random variable)

$$U(x) = \begin{cases} x, & x \leq k \\ k, & x \geq k \end{cases}$$

$s = 0$.

A decision-maker would never set $q > d(p)/(1 - 0.5\sigma)$ (i.e., all realizations of supply are greater than demand) nor $q < d(p)/(1 + 0.5\sigma)$ (i.e., all realizations of supply are less than demand). Similarly, a decision-maker would never set $p \leq c$ or $p \geq 1$ (i.e., $d(1) = 0$). For any $(p, q) \in \{(p, q) : p \in (c, 1), q \in (d(p)/(1 + 0.5\sigma), d(p)/(1 - 0.5\sigma))\}$,

$$\bar{S}(p,q) = p(1-p) - cq - \left(\frac{pq}{2\sigma}\right)\left(\frac{1-p}{q} - (1-0.5\sigma)\right)^2$$

$$\bar{S}_p = 1 - 2p - \frac{\left[1-p-q(1-0.5\sigma)\right]\left[1-3p-q(1-0.5\sigma)\right]}{2\sigma q}$$

$$\bar{S}_{pp} = -\frac{3p + 2q(1+0.5\sigma) - 2}{\sigma q} = -\frac{p + 2\left(q(1+0.5\sigma) - d(p)\right)}{\sigma q} < 0$$

$$\bar{S}_q = \left(\frac{p}{2\sigma}\right)\left[\left(\frac{1-p}{q}\right)^2 - (1-0.5\sigma)^2\right] - c$$

$$\bar{S}_{qq} = -\frac{p(1-p)^2}{\sigma q^3} < 0$$

$$\bar{S}_{pq} = \frac{1}{2\sigma}\left[\frac{(1-p)(1-3p)}{q^2} - (1-0.5\sigma)^2\right]$$

$$E\left[\left(d(p) - qY\right)^+\right] = \left(\frac{q}{2\sigma}\right)\left(\frac{1-p}{q} - (1-0.5\sigma)\right)^2$$

$$\frac{\partial}{\partial q}E\left[\left(d(p) - qY\right)^+\right] = \left(\frac{-1}{2\sigma}\right)\left[\left(\frac{1-p}{q}\right)^2 - (1-0.5\sigma)^2\right] \leq 0$$

$$\frac{\partial^2}{\partial p \partial q}E\left[\left(d(p) - qY\right)^+\right] = \frac{1-p}{\sigma q^2} \geq 0$$

$$\varepsilon(p, q) = \frac{2p(1-p)}{q^2\left[\left(\frac{1-p}{q}\right)^2 - (1-0.5\sigma)^2\right]}.$$

If $k \le pq(1-0.5\sigma) - cq$, then $S(p, q) = k$ for all realizations of Y. If $k \ge pd(p) - cq$, then $S(p, q) = \overline{S}(p,q)$. Therefore, we examine the case of $k \in (pq(1-0.5\sigma) - cq, pd(p) - cq)$; for any $(p, q) \in \{(p, q) : p \in (c, 1), q \in [d(p)/(1 + 0.5\sigma), d(p)/(1-0.5\sigma)\}$ and $k \in (pq(1-0.5\sigma) - cq, pd(p) - cq)$,

$$S(p, q) = \left(pqE\left[Y \mid Y \le \frac{k+cq}{pq}\right] - cq\right)\Phi_Y\left(\frac{k+cq}{pq}\right) + k\left(1 - \Phi_Y\left(\frac{k+cq}{pq}\right)\right)$$

$$= k - \left(\frac{pq}{2\sigma}\right)\left(\frac{k+cq}{pq} - (1-0.5\sigma)\right)^2$$

$$S_p = -\left(\frac{q}{2\sigma}\right)\left(\frac{k+cq}{pq} - (1-0.5\sigma)\right)^2 + \frac{2}{p}\left(\frac{pq}{2\sigma}\right)\left(\frac{k+cq}{pq} - (1-0.5\sigma)\right)\left(\frac{k+cq}{pq}\right)$$

$$= \left(\frac{1}{2\sigma}\right)\left[\frac{1}{p^2}\left(\frac{k^2}{q} + 2ck + c^2q\right) - q(1-0.5\sigma)^2\right]$$

$$S_{pp} = -\left(\frac{1}{\sigma}\right)\left(\frac{k^2}{p^3q} + \frac{2ck}{p^3} + \frac{c^2q}{p^3}\right) < 0$$

$$S_q = -\left(\frac{p}{2\sigma}\right)\left(\frac{k+cq}{pq} - (1-0.5\sigma)\right)^2 + 2\left(\frac{pq}{2\sigma}\right)\left(\frac{k+cq}{pq} - (1-0.5\sigma)\right)\frac{k}{pq^2}$$

$$= \left(\frac{1}{2\sigma}\right)\left[\frac{k^2}{pq^2} - \frac{c^2}{p} + 2c(1-0.5\sigma) - p(1-0.5\sigma)^2\right]$$

$$S_{qq} = -\left(\frac{1}{\sigma}\right)\left(\frac{k^2}{pq^3}\right) < 0$$

$$S_{pq} = -\left(\frac{1}{2\sigma}\right)\left[\left(\frac{k}{pq}\right)^2 - \left(\frac{c}{p}\right)^2 + (1-0.5\sigma)^2\right]$$

For our example instance, we set $\sigma = 1.8$, $c = 0.1$, $k = 0.01$, and $\mathbf{X} = \{(p, q) : p \in (0.2, 0.9), q \in ((1-p)/(1+\sigma/2), (1-p)/(1-\sigma/2))\}$. Note that $\mathbf{X}$ is a convex set. By applying the above expressions, it can be numerically verified that $\overline{S}_{pp} < 0$, $\overline{S}_{qq} < 0$, $\overline{S}_{pp}\overline{S}_{qq} - \overline{S}_{pq}{}^2 > 0$ for all $(p, q) \in \mathbf{X}$, and $\varepsilon(p, q) \ge 1/2$ for all $(p, q) \in \mathbf{X}$. Similarly, it can be numerically verified that there are points in $\mathbf{X}$ where $S_{pp}S_{qq} - S_{pq}{}^2 < 0$. As one example, at $p = 0.25$ and $q = 1.75$, we have $k = 0.01 \in (pq(1-0.5\sigma) - cq, pd(p) - cq) = (-0.131, 0.013)$ (i.e., $S(p, q) \ne \overline{S}(p,q)$ and $S(p, q) \ne k$), and thus we apply the above expressions to obtain

$$\bar{S}_{pp}\bar{S}_{qq} - \bar{S}_{pq}{}^2 = 0.025$$

$$S_{pp}S_{qq} - S_{pq}{}^2 = \text{-}0.002.$$

A.2 Single-Switch Properties

In this section we define two opposing properties, discuss the relationship between these properties and the single-crossing property, and present a result on the preservation of concavity of a multivariate function under risk aversion. We use this result in the proof of Proposition 1 that appears in the next section.

Lemma 3 in the next section shows that the risk-neutral objective function of P_S satisfies one of these properties and the risk-neutral objective function of P_D satisfies the other property. As discussed above, this structural difference between these two objective functions helps explain the differing results for P_S and P_D that appear in propositions 1 – 3.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ and let $F(\mathbf{x}, z)$ be a function defined on convex set $\mathbf{X} \times [z_L, z_H]$ that is nondecreasing in its last element for any $\mathbf{x} \in \mathbf{X}$. We say that the function satisfies a single-switch property (with respect to its last dimension) if the difference between a convex combination of the function evaluated at two points in $\mathbf{X}$ (for given z) and the function evaluated at the convex combination of the two points switches sign at most once as z increases over its interval $[z_L, z_H]$. This notion is formalized in the following definitions.

Increasing-Single-Switch Property (SS$^+$). *For any* $\mathbf{x}_1 \in \mathbf{X}$, $\mathbf{x}_2 \in \mathbf{X}$, *and* $\alpha \in (0, 1)$ *with* $\mathbf{x} = \alpha\mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2$, *if* $\alpha F(\mathbf{x}_1, z_1) + (1 - \alpha)F(\mathbf{x}_2, z_1) - F(\mathbf{x}, z_1) \geq 0$ *implies* $\alpha F(\mathbf{x}_1, z_1) + (1 - \alpha)F(\mathbf{x}_2, z_1) - F(\mathbf{x}, z_1) \geq 0$ *for all* $z \geq z_1$, *and if* $\alpha F(\mathbf{x}_1, z_1) + (1 - \alpha)F(\mathbf{x}_2, z_1) - F(\mathbf{x}, z_1) > 0$ *implies* $\alpha F(\mathbf{x}_1, z_1) + (1 - \alpha)F(\mathbf{x}_2, z_1) - F(\mathbf{x}, z_1) > 0$ *for all* $z \geq z_1$, *then* $F(\mathbf{x}, z)$ *satisfies the increasing-single-switch property.*

Decreasing-Single-Switch Property (SS$^-$). *For any* $\mathbf{x}_1 \in \mathbf{X}$, $\mathbf{x}_2 \in \mathbf{X}$, *and* $\alpha \in (0, 1)$ *with* $\mathbf{x} = \alpha\mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2$, *if* $\alpha F(\mathbf{x}_1, z_1) + (1 - \alpha)F(\mathbf{x}_2, z_1) - F(\mathbf{x}, z_1) \leq 0$ *implies* $\alpha F(\mathbf{x}_1, z_1) + (1 - \alpha)F(\mathbf{x}_2, z_1) - F(\mathbf{x}, z_1) \leq 0$ *for all* $z \geq z_1$, *and if* $\alpha F(\mathbf{x}_1, z_1) + (1 - \alpha)F(\mathbf{x}_2, z_1) - F(\mathbf{x}, z_1) < 0$ *implies* $\alpha F(\mathbf{x}_1, z_1) + (1 - \alpha)F(\mathbf{x}_2, z_1) - F(\mathbf{x}, z_1) < 0$ *for all* $z \geq z_1$, *then* $F(\mathbf{x}, z)$ *satisfies the decreasing-single-switch property.*

The single-switch properties are similar in spirit to the single-crossing property (SX) that is used in comparative statics (Milgrom and Shannon 1994). The function $F(x, z)$ satisfies SX if, for any $x_2 > x_1$, $F(x_2, z_1) - F(x_1, z_1) \geq 0$ implies $F(x_2, z) - F(x_1, z) \geq 0$ for all $z \geq z_1$, and $F(x_2, z_1) - F(x_1, z_1) > 0$ implies $F(x_2, z) - F(x_1, z) > 0$ for all $z \geq z_1$. A function $F(x, z)$ that satisfies SS$^+$ exhibits similar character to a function with partial derivative $F_x(x, z)$ that satisfies SX, as shown below.

Lemma 1. *$F(x, z)$ is twice-differentiable with respect to x. If F satisfies* SS^+ *or if F_x satisfies* SX*, then $F_{xx}(x, z_1) \geq 0$ implies $F_{xx}(x, z) \geq 0$ for all $z \geq z_1$, and $F_{xx}(x, z_1) > 0$ implies $F_{xx}(x, z) > 0$ for all $z \geq z_1$.*

Proof. Let h be a positive number. Because F satisfies SS^+, if

$$0.5F(x, z_1) + 0.5F(x + 2h, z_1) - F(x + h, z_1) \geq 0$$

then

$$0.5F(x, z) + 0.5F(x + 2h, z) - F(x + h, z) \geq 0 \text{ for all } z \geq z_1.$$

Thus, if

$$\lim_{h\to 0}\frac{F(x+2h,z_1)-2F(x+h,z_1)+F(x,z_1)}{h^2}=F_{xx}(x,z_1)\geq 0$$

then

$$\lim_{h\to 0}\frac{F(x+2h,z)-2F(x+h,z)+F(x,z)}{h^2}=F_{xx}(x,z) \text{ for any } z \geq z_1.$$

Following the same steps with strict inequalities, $F_{xx}(x, z_1) > 0$ implies $F_{xx}(x, z) > 0$ for any $z \geq z_1$.

Because F_x satisfies SX, if

$$F_x(x + h, z_1) - F_x(x, z_1) \geq 0,$$

then

$$F_x(x + h, z) - F_x(x, z) \geq 0 \text{ for any } z \geq z_1,$$

Thus, if

$$\lim_{h\to 0}\frac{F_x(x+h,z_1)-F_x(x,z_1)}{h}=F_{xx}(x,z_1)\geq 0$$

then

$$\lim_{h\to 0}\frac{F_x(x+h,z)-F_x(x,z)}{h}=F_{xx}(x,z)\geq 0 \text{ for any } z \geq z_1.$$

Following the same steps with strict inequalities, $F_{xx}(x, z_1) > 0$ implies $F_{xx}(x, z) > 0$ for any $z \geq z_1$. ❒

Lemma 2. *If $E[F(\mathbf{x},Z)]$ is concave on $\mathbf{X}$ and $F(\mathbf{x},Z)$ satisfies* SS^+*, then $E[U(F(\mathbf{x},Z))]$ is concave on $\mathbf{X}$.*

Proof. Let $\phi(z)$ denote the pdf of Z with support $[z_L, z_H]$. For a given $\alpha \in (0, 1)$, $\mathbf{x}_1 \in \mathbf{X}$, and $\mathbf{x}_2 \in \mathbf{X}$, let $z^* = \max\{z \mid \alpha F(\mathbf{x}_1, z) + (1 - \alpha)F(\mathbf{x}_2, z) - F(\mathbf{x}, z) \leq 0, z \in [z_L, z_H]\}$, $b(z) = \alpha F(\mathbf{x}_1, z) + (1 - \alpha)F(\mathbf{x}_2, z)$, $a(z) = F(\mathbf{x}, z)$, $H = \alpha E[F(\mathbf{x}_1, Z)] + (1 - \alpha)E[F(\mathbf{x}_2, Z)] - E[F(\mathbf{x}, Z)]$, and $K = \alpha E[U(F(\mathbf{x}_1, Z))] + (1 - \alpha)E[U(F(\mathbf{x}_2, Z))] - E[U(F(\mathbf{x}, Z))]$. Since $E[F(\mathbf{x}, Z)]$ is concave, we have

$$H \leq 0. \tag{4}$$

From the fact that U is a concave increasing function, it follows that for $y_1 > y_2$,

$$U(y_1) - U(y_2) \geq U'(y)(y_1 - y_2)\ \forall y \geq y_1 \text{ and } U(y_1) - U(y_2) \leq U'(y)(y_1 - y_2)\ \forall y \leq y_2. \tag{5}$$

We are now ready to show that $H \le 0$ implies $K \le 0$:

$$K = \int_\Omega \left[\alpha U\left(F\left(\mathbf{x}_1, z\right)\right) + (1-\alpha) U\left(F\left(\mathbf{x}_2, z\right)\right)\right]\phi(z)\,dz - \int_\Omega U\left(F\left(\mathbf{x}, z\right)\right)\phi(z)\,dz$$

$$\le \int_\Omega U\left(\alpha F\left(\mathbf{x}_1, z\right) + (1-\alpha) F\left(\mathbf{x}_2, z\right)\right)\phi(z)\,dz - \int_\Omega U\left(F\left(\mathbf{x}, z\right)\right)\phi(z)\,dz \qquad \text{(due to concave } U\text{)}$$

$$= \int_{z>z^*} \left[U\left(b(z)\right) - U\left(a(z)\right)\right]\phi(z)\,dz - \int_{z \le z^*} \left[U\left(a(z)\right) - U\left(b(z)\right)\right]\phi(z)\,dz$$

(where each integrand is nonnegative due to SS^+)

$$\le U'\left(a\left(z^*\right)\right) \int_{z>z^*} \left(b(z) - a(z)\right)\phi(z)\,dz - U'\left(a\left(z^*\right)\right) \int_{z \le z^*} \left(a(z) - b(z)\right)\phi(z)\,dz \qquad \text{(due to } F_z(\mathbf{x}, z) \ge 0\text{)}$$

$$= U'\left(a\left(z^*\right)\right) H$$

$$\le 0, \qquad \text{(due to } U' > 0\text{)}$$

i.e., $K \le 0$ for any given $\alpha \in (0, 1)$, $\mathbf{x}_1 \in \mathbf{X}$, and $\mathbf{x}_2 \in \mathbf{X}$, and thus $E[U(F(\mathbf{x}, Z))]$ is concave. ❒

A.3 Proofs

We begin by presenting Proposition 4 that pertains to a problem that includes both supply and demand uncertainty. This is followed by Corollary 1, Lemma 3, and Lemma 4. A special case of Proposition 4 yields one of the results given in Proposition 1. Corollary 1 is referred to in the discussion that follows Proposition 1 in the body of the manuscript. Lemma 3 shows that the expected profit functions of P_S and P_D satisfy opposing single-switch properties; we use this result in the proof of Proposition 1. Lemma 4 is used in the proof of Proposition 2. After these results, we present the proofs of propositions 1 – 3.

We define the joint pdf and cdf of random variables Y and Z are $\phi(y, z)$ and $\Phi(y, z)$. Recall that the marginal density and distribution functions are $\phi_i(\cdot)$ and $\Phi_i(\cdot)$ for $i \in \{Y, Z\}$. The random profit under uncertain supply and demand is denoted $\tilde{R}\left(p, q, Y, Z\right) = p \min\left\{d\left(p, Z\right), qY\right\} - cq + sq\left(Y - 1\right)$ and the expected utility is denoted $R(p, q) = E\left[U\left(\tilde{R}\left(p, q, Y, Z\right)\right)\right]$ (the expectation operator is applied to all random variables within its brackets, in this case, both Y and Z). Functions $S(p, q)$ and $D(p, q)$ are special cases of $R(p, q)$. We use r to denote the revenue function in this section, i.e., $r(p) = pd(p)$ for P_S and $r(p, z) = pd(p, z)$ for P_D.

Proposition 4. *If* $\varepsilon(p, q)u(p, q)/u^+(p, q) \ge ½$ *for all* $(p, q) \in \mathbf{X}$, *then* $R(p, q)$ *is concave*.

If the newsvendor is risk neutral, then $u(p, q)/u^+(p, q) = 1$ for all $(p, q) \in \mathbf{X}$, which leads to the following corollary.

Corollary 1. *If the newsvendor is risk neutral and* $\varepsilon(p, q) \geq ½$ *for all* $(p, q) \in \mathbf{X}$, *then* $S(p, q)$, $D(p, q)$, *and* $R(p, q)$ *are concave*.

Proof of Proposition 4. We develop the proof with random variable Y conditioned on the realization of Z, which by setting Z to be deterministic, leads to results for P_S. The key to the proof is exposing a special structure in the determinant of the Hessian of $R(p, q)$ and the application of different forms of the Schwarz inequality. We obtain this structure via substitution after deriving several identities. The structure contains four major elements. The sign of the first two elements have a sign that follows from inequalities derived earlier in the proof. The sign of the third element becomes apparent through application of a continuous form of the Schwarz inequality. The fourth element contains a term that can be both negative and positive (depending on parameter values). However, by application of the discrete form of the Schwarz inequality in combination with known properties of the third element along with inequalities obtained earlier in the proof, we are able to guarantee that the fourth element is nonnegative regardless of the sign of this particular term. A number of expressions required for the determinant of the Hessian are long, so we introduce some new notation, thereby obtaining expressions in a more compact form than would otherwise be possible.

Let $\tilde{R}^{-}(p,q,y,z) = r(p, z) - cq + sq(y - 1)$ (i.e., profit realizations where demand is less than supply) and $\tilde{R}^{+}(p,q,y,z) = pqy - cq + sq(y - 1)$ (i.e., profit realizations where demand is more than supply).

$$\tilde{R}^{-}(p,q,d(p,z)/q,z) = \tilde{R}^{+}(p,q,d(p,z)/q,z) = r(p, z) - cq + s(d(p, z) - q)$$

$$\tilde{R}^{-}_{pp}(p,q,y,z) = r_{pp}(p, z)$$

$$\tilde{R}^{+}_{pp}(p,q,y,z) = 0$$

$$\tilde{R}^{+}_{q}(p,q,d(p,z)/q,z) - \tilde{R}^{-}_{q}(p,q,d(p,z)/q,z) = pd(p, z)/q$$

$$\tilde{R}_{qq}(p,q,y,z) = 0$$

$$\tilde{R}^{-}_{pq}(p,q,y,z) = 0$$

$$\tilde{R}^{+}_{pq}(p,q,y,z) = y$$

$$R_p(p, q \mid z) \equiv E\left[\partial U\left(\tilde{R}(p,q,Y,z)\right)/\partial p\right] = E\left[U'\left(\tilde{R}(p,q,Y,z)\right)\tilde{R}_p(p,q,Y,z)\right]$$

$$R_q(p, q \mid z) \equiv E\left[\partial U\left(\tilde{R}(p,q,Y,z)\right)/\partial q\right] = E\left[U'\left(\tilde{R}(p,q,Y,z)\right)\tilde{R}_q(p,q,Y,z)\right]$$

$$R_p(p, q) = E\left[U'\left(\tilde{R}(p,q,Y,Z)\right)\tilde{R}_p(p,q,Y,Z)\right]$$

$$R_q(p, q) = E\left[U'\left(\tilde{R}(p,q,Y,Z)\right)\tilde{R}_q(p,q,Y,Z)\right]$$

$z_p(p, d) > 0$ (6)

$z_x(p, d) > 0$ (7)

(the last two inequalities can be obtained from $d_p(p, z) < 0$, $d_z(p, z) > 0$, and implicit differentiation of $d(p, z(p, d)) = d$).

The Arrow-Pratt measure of absolute risk aversion at wealth x is $\gamma(x) = -U''(x)/U'(x)$. Beginning with $R_p(p, q \mid z)$, we obtain the cross-derivative

$$R_{pq}(p, q \mid z) = E\left[U'\left(\tilde{R}(p,q,Y,z)\right)\begin{pmatrix}\tilde{R}_{pq}(p,q,Y,z) - \\ \gamma\left(\tilde{R}(p,q,Y,z)\right)\tilde{R}_p(p,q,Y,z)\tilde{R}_q(p,q,Y,z)\end{pmatrix}\right] -$$

$$U'\left(\tilde{R}\left(p,q,\frac{d(p,z)}{q},z\right)\right)\begin{pmatrix}\tilde{R}_p^+\left(p,q,\frac{d(p,z)}{q},z\right) - \\ \tilde{R}_p^-\left(p,q,\frac{d(p,z)}{q},z\right)\end{pmatrix}\phi\left(\frac{d(p,z)}{q},z\right)\frac{d(p,z)}{q^2}$$

and beginning with $R_q(p, q \mid z)$, we obtain the cross-derivative

$$R_{pq}(p, q \mid z) = E\left[U'\left(\tilde{R}(p,q,Y,z)\right)\begin{pmatrix}\tilde{R}_{pq}(p,q,Y,z) - \\ \gamma\left(\tilde{R}(p,q,Y,z)\right)\tilde{R}_p(p,q,Y,z)\tilde{R}_q(p,q,Y,z)\end{pmatrix}\right] +$$

$$U'\left(\tilde{R}\left(p,q,\frac{d(p,z)}{q},z\right)\right)\begin{pmatrix}\tilde{R}_q^+\left(p,q,\frac{d(p,z)}{q},z\right) - \\ \tilde{R}_q^-\left(p,q,\frac{d(p,z)}{q},z\right)\end{pmatrix}\phi\left(\frac{d(p,z)}{q},z\right)\frac{d_p(p,z)}{q}$$

which implies

$$\tilde{R}_p^+\left(p,q,\frac{d(p,z)}{q},z\right) - \tilde{R}_p^-\left(p,q,\frac{d(p,z)}{q},z\right) = -\begin{pmatrix}\tilde{R}_q^+\left(p,q,\frac{d(p,z)}{q},z\right) - \\ \tilde{R}_q^-\left(p,q,\frac{d(p,z)}{q},z\right)\end{pmatrix}\frac{q d_p(p,z)}{d(p,z)}. \quad (8)$$

$$R_{pp}(p, q \mid z) = E\left[U'\left(\tilde{R}(p,q,Y,z)\right)\begin{pmatrix}\tilde{R}_{pp}(p,q,Y,z) - \\ \gamma\left(\tilde{R}(p,q,Y,z)\right)\tilde{R}_p(p,q,Y,z)^2\end{pmatrix}\right] +$$

$$U'\left(\tilde{R}\left(p,q,\frac{d(p,z)}{q},z\right)\right)\begin{pmatrix}\tilde{R}_p^+\left(p,q,\frac{d(p,z)}{q},z\right) - \\ \tilde{R}_p^-\left(p,q,\frac{d(p,z)}{q},z\right)\end{pmatrix}\phi\left(\frac{d(p,z)}{q},z\right)\frac{d_p(p,z)}{q}$$

$$= E\left[U'\left(\tilde{R}(p,q,Y,z)\right)\begin{pmatrix}\tilde{R}_{pp}(p,q,Y,z) - \\ \gamma\left(\tilde{R}(p,q,Y,z)\right)\tilde{R}_p(p,q,Y,z)^2\end{pmatrix}\right] -$$

$$U'\left(\tilde{R}\left(p,q,\frac{d(p,z)}{q},z\right)\right)\begin{pmatrix}\tilde{R}_q^+\left(p,q,\frac{d(p,z)}{q},z\right) - \\ \tilde{R}_q^-\left(p,q,\frac{d(p,z)}{q},z\right)\end{pmatrix}\phi\left(\frac{d(p,z)}{q},z\right)\frac{d_p(p,z)^2}{d(p,z)}$$

(see (8))

$$R_{qq}(p, q \mid z) = E\left[-U'\left(\tilde{R}(p,q,Y,z)\right)\gamma\left(\tilde{R}(p,q,Y,z)\right)\tilde{R}_q(p,q,Y,z)^2\right] -$$

$$U'\left(\tilde{R}\left(p,q,\frac{d(p,z)}{q},z\right)\right)\begin{pmatrix}\tilde{R}_q^+\left(p,q,\frac{d(p,z)}{q},z\right) - \\ \tilde{R}_q^-\left(p,q,\frac{d(p,z)}{q},z\right)\end{pmatrix}\phi\left(\frac{d(p,z)}{q},z\right)\frac{d(p,z)}{q^2}$$

We make use of the following notation and inequalities.

$$B(p, q, z) = \left(\tilde{R}_q^+\left(p,q,\frac{d(p,z)}{q},z\right) - \tilde{R}_q^-\left(p,q,\frac{d(p,z)}{q},z\right)\right)\phi\left(\frac{d(p,z)}{q},z\right)\frac{d(p,z)}{q^2}$$

$$= \phi\left(\frac{d(p,z)}{q},z\right)\frac{pd(p,z)^2}{q^3} \geq 0$$

$\Omega^-(z) = \left(Y \geq d(p,z)/q\right)$ = event that demand is not more than supply

$\Omega^+(z) = \left(Y < d(p,z)/q\right)$ = event that demand is more than supply

$$\varepsilon(p, q) = \frac{-p\frac{\partial^2}{\partial p\partial q}E\left[\left(d(p,Z) - qY\right)^+\right]}{\frac{\partial}{\partial q}E\left[\left(d(p,Z) - qY\right)^+\right]} = \frac{E\left[B(p,q,Z)\left(-qd_p(p,Z)/d(p,Z)\right)\right]}{E\left[Y;\Omega^+(Z)\right]} \geq 0$$

$$u(p, q) = \frac{E\left[U'\left(r(p,Z) - cq + s\left(d(p,Z) - q\right)\right)B(p,q,Z)\left(qd_p(p,Z)/d(p,Z)\right)\right]}{E\left[B(p,q,Z)\left(qd_p(p,Z)/d(p,Z)\right)\right]}$$

$$= \int_{z_L}^{z_H} U'\left(r(p,z) - cq + s\left(d(p,z) - q\right)\right)\left(\frac{B(p,q,z)\left(qd_p(p,z)/d(p,z)\right)\phi_Z(z)}{\int_{z_L}^{z_H} B(p,q,t)\left(qd_p(p,t)/d(p,t)\right)\phi_Z(t)\,dt}\right)dz$$

$$u^{-}(p,q)=\frac{E\left[U'\left(\tilde{R}(p,q,Y,Z)\right)\tilde{R}_{pp}(p,q,Y,Z)\right]}{E\left[\tilde{R}_{pp}(p,q,Y,Z)\right]}$$

$$=\int_{z_L}^{z_H}\int_{d(p,z)/q}^{y_H}U'\left(r(p,z)-cq+sq(y-1)\right)\left(\frac{r_{pp}(p,z)\phi(y,z)}{\int_{z_L}^{z_H}\int_{d(p,t)/q}^{y_H}r_{pp}(p,s)\phi(s,t)\,dsdt}\right)dydz\geq 0$$

$$u^{+}(p,q)=\frac{E\left[U'\left(\tilde{R}(p,q,Y,Z)\right)\tilde{R}_{pq}(p,q,Y,Z)\right]}{E\left[\tilde{R}_{pq}(p,q,Y,Z)\right]}=\frac{E\left[U'\left(\tilde{R}(p,q,Y,Z)\right)Y;\Omega^{+}(Z)\right]}{E\left[Y;\Omega^{+}(Z)\right]}$$

$$=\int_{z_L}^{z_H}\int_{y_L}^{d(p,z)/q}U'\left(\tilde{R}^{+}(p,q,y,z)\right)\left(\frac{y\phi(y,z)}{\int_{z_L}^{z_H}\int_{y_L}^{d(p,t)/q}s\phi(s,t)\,dsdt}\right)dydz\geq 0$$

$$K1(p,q)=E\left[U'\left(\tilde{R}(p,q,Y,Z)\right)\gamma\left(\tilde{R}(p,q,Y,Z)\right)\tilde{R}_{p}(p,q,Y,Z)^{2}\right]\geq 0$$

$$K2(p,q)=E\left[U'\left(\tilde{R}(p,q,Y,Z)\right)\gamma\left(\tilde{R}(p,q,Y,Z)\right)\tilde{R}_{q}(p,q,Y,Z)^{2}\right]\geq 0$$

$$K3(p,q)=E\left[U'\left(\tilde{R}(p,q,Y,Z)\right)\gamma\left(\tilde{R}(p,q,Y,Z)\right)\tilde{R}_{p}(p,q,Y,Z)\tilde{R}_{q}(p,q,Y,Z)\right]$$

$$L1(p,q)=E\left[U'\begin{pmatrix}r(p,Z)-cq+\\ s\left(d(p,Z)-q\right)\end{pmatrix}\begin{pmatrix}\tilde{R}_{q}^{+}\left(p,q,\frac{d(p,Z)}{q},Z\right)-\\ \tilde{R}_{q}^{-}\left(p,q,\frac{d(p,Z)}{q},Z\right)\end{pmatrix}\phi\left(\frac{d(p,Z)}{q},Z\right)\frac{d_{p}(p,Z)^{2}}{d(p,Z)}\right]$$

$$=E\left[U'\begin{pmatrix}r(p,Z)-cq+\\ s\left(d(p,Z)-q\right)\end{pmatrix}B(p,q,Z)\left(\frac{qd_{p}(p,Z)}{d(p,Z)}\right)^{2}\right]\geq 0$$

$$L2(p,q)=E\left[U'\begin{pmatrix}r(p,Z)-cq+\\ s\left(d(p,Z)-q\right)\end{pmatrix}\begin{pmatrix}\tilde{R}_{q}^{+}\left(p,q,\frac{d(p,Z)}{q},Z\right)-\\ \tilde{R}_{q}^{-}\left(p,q,\frac{d(p,Z)}{q},Z\right)\end{pmatrix}\phi\left(\frac{d(p,Z)}{q},Z\right)\frac{d(p,Z)}{q^{2}}\right]$$

$$=E\left[U'\begin{pmatrix}r(p,Z)-cq+\\ s\left(d(p,Z)-q\right)\end{pmatrix}B(p,q,Z)\right]\geq 0$$

$$L3(p,q) = E\left[-U'\begin{pmatrix} r(p,Z) - cq + \\ s(d(p,Z) - q) \end{pmatrix} \begin{pmatrix} \tilde{R}_q^{+}\left(p,q,\frac{d(p,Z)}{q},Z\right) - \\ \tilde{R}_q^{-}\left(p,q,\frac{d(p,Z)}{q},Z\right) \end{pmatrix} \phi\left(\frac{d(p,Z)}{q},Z\right)\frac{d_p(p,Z)}{q} \right]$$

$$= E\left[U'\begin{pmatrix} r(p,Z) - cq + \\ s(d(p,Z) - q) \end{pmatrix} B(p,q,Z)\left(\frac{-qd_p(p,Z)}{d(p,Z)}\right) \right]$$

$$= u(p,q)E\left[Y;\Omega^{+}(Z)\right]\varepsilon(p,q) \geq 0. \tag{9}$$

Using the preceding notation, we get

$$R_{pp}(p,q) = E\left[R_{pp}(p,q \mid Z)\right]$$

$$= E\left[U'\left(\tilde{R}(p,q,Y,Z)\right)\begin{pmatrix} \tilde{R}_{pp}(p,q,Y,Z) - \\ \gamma\left(\tilde{R}(p,q,Y,Z)\right)\tilde{R}_p(p,q,Y,Z)^2 \end{pmatrix} \right] -$$

$$E\left[U'\begin{pmatrix} r(p,Z) - cq + \\ s(d(p,Z) - q) \end{pmatrix} \begin{pmatrix} \tilde{R}_q^{+}\left(p,q,\frac{d(p,Z)}{q},Z\right) - \\ \tilde{R}_q^{-}\left(p,q,\frac{d(p,Z)}{q},Z\right) \end{pmatrix} \phi\left(\frac{d(p,Z)}{q},Z\right)\frac{d_p(p,Z)^2}{d(p,Z)} \right]$$

$$= u^{-}(p,q)E\left[r_{pp}(p,Z);\Omega^{-}(Z)\right] - K1(p,q) - L1(p,q) < 0 \tag{10}$$

$$R_{qq}(p,q) = E\left[R_{qq}(p,q \mid Z)\right]$$

$$= E\left[-U'\left(\tilde{R}(p,q,Y,Z)\right)\gamma\left(\tilde{R}(p,q,Y,Z)\right)\tilde{R}_q(p,q,Y,Z)^2\right] -$$

$$E\left[U'\begin{pmatrix} r(p,Z) - cq + \\ s(d(p,Z) - q) \end{pmatrix} \begin{pmatrix} \tilde{R}_q^{+}\left(p,q,\frac{d(p,Z)}{q},Z\right) - \\ \tilde{R}_q^{-}\left(p,q,\frac{d(p,Z)}{q},Z\right) \end{pmatrix} \phi\left(\frac{d(p,Z)}{q},Z\right)\frac{d(p,Z)}{q^2} \right]$$

$$= -K2(p,q) - L2(p,q) < 0 \tag{11}$$

$$R_{pq}(p,q) = E\left[R_{pq}(p,q \mid Z)\right]$$

$$= E\left[U'\left(\tilde{R}(p,q,Y,Z)\right)\begin{pmatrix} \tilde{R}_{pq}(p,q,Y,Z) - \\ \gamma\left(R(p,q,Y,Z)\right)\tilde{R}_p(p,q,Y,Z)\tilde{R}_q(p,q,Y,Z) \end{pmatrix} \right] +$$

$$E\left[U'\begin{pmatrix} r(p,Z)-cq+ \\ s(d(p,Z)-q)\end{pmatrix}\begin{pmatrix}\tilde{R}_q^+\left(p,q,\frac{d(p,Z)}{q},Z\right)- \\ \tilde{R}_q^-\left(p,q,\frac{d(p,Z)}{q},Z\right)\end{pmatrix}\phi\left(\frac{d(p,Z)}{q},Z\right)\frac{d_p(p,Z)}{q}\right]$$

$$=u^+(p,q)E\left[Y;\Omega^+(Z)\right]-K3(p,q)-L3(p,q)$$

$$=u^+(p,q)E\left[Y;\Omega^+(Z)\right]-K3(p,q)-u(p,q)E\left[Y;\Omega^+(Z)\right]\varepsilon(p,q)$$

and the determinant of the Hessian is

$$\Delta = R_{pp}(p, q)R_{qq}(p, q) - R_{pq}(p, q)^2$$

$$=u^-(p,q)E\left[-r_{pp}(p,Z);\Omega^-(Z)\right]\left(K2(p,q)+L2(p,q)\right)+$$

$$\left(u^+(p,q)E\left[Y;\Omega^+(Z)\right]\right)^2\left(2\varepsilon(p,q)\frac{u(p,q)}{u^+(p,q)}-1\right)+$$

$$\left(K1(p,q)K2(p,q)-K3(p,q)^2\right)+\left(L1(p,q)L2(p,q)-L3(p,q)^2\right)+$$

$$K1(p,q)L2(p,q)+K2(p,q)L1(p,q)+2K3(p,q)\left(u^+(p,q)E\left[Y;\Omega^+(Z)\right]-L3(p,q)\right)$$

Suppose that

$$\varepsilon(p,q)\frac{u(p,q)}{u^+(p,q)}\geq\frac{1}{2}. \tag{12}$$

The first line is nonnegative because it is the product of two nonnegative terms. The second line is nonnegative due to (12). Each of the parenthetical terms in the third line is nonnegative due to the Schwarz inequality, which for double and single integration is

$$\left[\iint_A f(y,z)^2\,dydz\right]\left[\iint_A g(y,z)^2\,dydz\right]\geq\left[\iint_A f(y,z)\,f(y,z)\,dydz\right]^2$$

$$\left[\int_a^b f(z)^2\,dz\right]\left[\int_a^b g(z)^2\,dz\right]\geq\left[\int_a^b f(z)g(z)\,dz\right]^2$$

In the case of the first parenthetical term, $K1(p, q)K2(p, q) - K3(p, q)^2$,

$$f(y,z)=\left(U'\left(\tilde{R}(p,q,y,z)\right)\gamma\left(\tilde{R}(p,q,y,z)\right)\phi(y,z)\right)^{1/2}\tilde{R}_p(p,q,y,z)$$

$$g(y,z)=\left(U'\left(\tilde{R}(p,q,y,z)\right)\gamma\left(\tilde{R}(p,q,y,z)\right)\phi(y,z)\right)^{1/2}\tilde{R}_q(p,q,y,z)$$

and in the case of the second parenthetical term, $L1(p, q)L2(p, q) - L3(p, q)^2$,

$$f(z)=\left(U'\left(r(p,z)-cq+s\left(d(p,z)-q\right)\right)B(p,q,z)\phi_Z(z)\right)^{1/2}\left(-qd_p(p,z)/d(p,z)\right)$$

$$g(z) = \left(U'\left(r(p,z) - cq + s\left(d(p,z) - q\right)\right) B(p,q,z) \phi_Z(z)\right)^{1/2} .$$

Let us now consider the fourth line. The only term with indeterminate sign in the fourth line is $K3(p, q)$ (all of the other terms are nonnegative). If $K3(p, q) = 0$, then the fourth line is clearly nonnegative. Suppose that $K3(p, q) \neq 0$. To simplify the notation, we suppress p, q, and Z, and the fourth line is

$$K = K1L2 + K2L1 + 2K3\left(u^+ E\left[Y;\Omega^+\right] - L3\right).$$

Case 1. $K3 > 0$: A discrete version of the Schwarz inequality is

$$\left(a_1^2 + a_2^2\right)\left(b_1^2 + b_2^2\right) \geq \left(a_1 b_1 + a_2 b_2\right)^2.$$

Therefore

$$\begin{aligned} K1K2 + L1L2 + K1L2 + K2L1 &= \left(\left(\sqrt{K1}\right)^2 + \left(\sqrt{L1}\right)^2\right)\left(\left(\sqrt{K2}\right)^2 + \left(\sqrt{L2}\right)^2\right) \\ &\geq \left(\sqrt{K1K2} + \sqrt{L1L2}\right)^2 \\ &\geq K1K2 + L1L2 + 2\sqrt{K1K2L1L2} \end{aligned}$$

Recall from analysis of the third line that $K1K2 \geq K3^2$ and $L1L2 \geq L3^2$. Substituting these inequalities into the above and simplifying, we get

$$K1L2 + K2L1 - 2K3L3 \geq 0,$$

which implies

$$K = K1L2 + K2L1 - 2K3L3 + 2K3u^+ E\left[Y;\Omega^+\right] \geq 0.$$

Case 2. $K3 < 0$: We rewrite the fourth line as

$$K = K1L2 + K2L1 - 2(-K3)u^+ E\left[Y;\Omega^+\right]\left(1 - \varepsilon\frac{u}{u^+}\right).$$

From (12) it follows that

$$\begin{aligned} K &\geq K1L2 + K2L1 - (-K3)u^+ E\left[Y;\Omega^+\right] \\ &= K1L2 + K2L1 - (-K3)\frac{L3}{\varepsilon\left(u/u^+\right)} && \text{(due to } L3 = uE\left[Y;\Omega^+\right]\varepsilon\text{; see (9))} \\ &\geq K1L2 + K2L1 - 2(-K3)L3 && \text{(due to (12))} \\ &\geq 0 && \text{(see Case 1).} \end{aligned}$$

Note that $R_{pp} \leq 0$ and $R_{qq} \leq 0$ (see above). Thus, (12) assures that R is concave. ❒

Lemma 3. *(a)* $\tilde{S}(p,q,Y)$ *satisfies* SS$^-$. *(b)* $\tilde{D}(p,q,Z)$ *satisfies* SS$^+$.

Proof. Part (*a*): The realized profit function of P_S is $\tilde{S}(p,q,y) = p\min\{d(p), qy\} - cq + sq(y-1)$. For a given $\alpha \in (0, 1)$, $(p_1, q_1) \in \mathbf{X}$, $(p_2, q_2) \in \mathbf{X}$, with $(p, q) = (\alpha p_1 + (1-\alpha)p_2, \alpha q_1 + (1-\alpha)q_2)$, let

$$g(y) = \alpha p_1\min\{d(p_1), q_1y\} + (1-\alpha)p_2\min\{d(p_2), q_2y\} - p\min\{d(p), qy\}.$$

We must show that $g(y)$ switches from positive to negative no more than once as y increases over interval $[y_L, y_H]$. In order to do this, we consider all possible cases regarding inequalities among parameters.

Without loss of generality we index prices such that $p_1 < p_2$, and we let $\delta_p = p_2 - p_1$ and $\delta_q = |q_2 - q_1|$. We refer to the case of $\delta_q = q_1 - q_2$ (i.e., $q_1 > q_2$) as Scenario 1 (S1) and we refer to the case of $\delta_q = q_2 - q_1$ (i.e., $q_2 > q_1$) as Scenario 2 (S2). We can write p_i and q_i in terms of p, q, α, δ_p, and δ_q for each scenario.

S1: $(p_1, q_1) = (p - (1-\alpha)\delta_p, q + (1-\alpha)\delta_q)$, $(p_2, q_2) = (p + \alpha\delta_p, q - \alpha\delta_q)$

S2: $(p_1, q_1) = (p - (1-\alpha)\delta_p, q - (1-\alpha)\delta_q)$, $(p_2, q_2) = (p + \alpha\delta_p, q + \alpha\delta_q)$

For a given p, q, α, δ_p, and δ_q, which determine p_1, q_1, p_2, q_2, we define

$y_0 = d(p)/q$,

$y_1 = d(p_1)/q_1$,

$y_2 = d(p_2)/q_2$.

Note that (p_1, q_1) and (p_2, q_2) (and thus (p, q)) are elements of $\mathbf{X}$ (otherwise the points are not relevant for assessing the single-switch property on $\mathbf{X}$), and thus $\min\{y_0, y_1, y_2\} \ge 0$. We consider how the sign of $g(y)$ changes for values of $0 \le y < \min\{y_0, y_1, y_2\}$ to $y > \max\{y_0, y_1, y_2\}$, which is assured to span the support $[y_L, y_H]$. Note that

$$g(y) = \alpha p_1 d(p_1) + (1-\alpha)p_2 d(p_2) - pd(p) \le 0 \quad (13)$$

(due to the strict concavity of $pd(p)$).

We begin with S1, for which $p_1 < p < p_2$,

$d(p_1) > d(p) > d(p_2)$

$q_1 > q > q_2$.

For $0 \le y \le \min\{y_0, y_1, y_2\}$,

$$\begin{aligned} g(y) &= [\alpha p_1q_1 + (1-\alpha)p_2q_2 - pq]y \\ &= \{\alpha[p - (1-\alpha)\delta_p][q + (1-\alpha)\delta_q] + (1-\alpha)[p + \alpha\delta_p][q - \alpha\delta_q]\}y - pqy \\ &= -\alpha(1-\alpha)\delta_p\delta_q y \le 0. \end{aligned} \quad (14)$$

For $y \ge \max\{y_0, y_1, y_2\}$,

$$g(y) = \alpha p_1 d(p_1) + (1-\alpha)p_2 d(p_2) - pd(p) \le 0$$

(see (13)).

We consider three possibilities for the ordering of y_0, y_1, y_2, and for each possibility we examine $g(y)$ for $y \in (\min\{y_0, y_1, y_2\}, \max\{y_0, y_1, y_2\}]$. We will see that $g(y) \ge 0$ for all y in this range.

Suppose that $y_0 < y_1 \leq y_2$ (the analysis is same for the case of $y_0 < y_2 \leq y_1$). For $y \in (y_0, y_1]$,

$$g(y) = [\alpha p_1 q_1 + (1 - \alpha) p_2 q_2] y - p d(p)$$

$$= \alpha p_1 d(p_1) + (1 - \alpha) p_2 d(p_2) - \alpha p_1 [d(p_1) - q_1 y] - (1 - \alpha) p_2 [d(p_2) - q_2 y] - p d(p) < 0$$

(due to (13), $d(p_1) \geq q_1 y$, and $d(p_2) > q_2 y$), and for $y \in (y_1, y_2]$,

$$g(y) = \alpha p_1 d(p_1) + (1 - \alpha) p_2 q_2 y - p d(p)$$

$$= \alpha p_1 d(p_1) + (1 - \alpha) p_2 d(p_2) - (1 - \alpha) p_2 [d(p_2) - q_2 y] - p d(p) \leq 0$$

(due to (13) and $d(p_2) \geq q_2 y$). Thus, $g(y) \leq 0$ for all $y \in [y_L, y_H]$ (i.e., SS^- holds).

Suppose that $y_1 < y_0 \leq y_2$ (the analysis is same for the case of $y_2 < y_0 \leq y_1$). For $y \in (y_1, y_0]$,

$$g(y) = \alpha p_1 d(p_1) + (1 - \alpha) p_2 q_2 y - p q y$$

$$= -\{[pq - (\alpha p_1 q_1 + (1 - \alpha) p_2 q_2)] y + \alpha p_1 [q_1 y - d(p_1)]\} < 0$$

(due to (14) and $q_1 y > d(p_1)$), and for $y \in (y_0, y_2]$,

$$g(y) = \alpha p_1 d(p_1) + (1 - \alpha) p_2 q_2 y - p d(p)$$

$$= \alpha p_1 d(p_1) + (1 - \alpha) p_2 d(p_2) - (1 - \alpha) p_2 [d(p_2) - q_2 y] - p d(p) \leq 0$$

(due to (13) and $d(p_2) \geq q_2 y$). Thus, $g(y) \leq 0$ for all $y \in [y_L, y_H]$ (i.e., SS^- holds).

Finally, suppose that $y_1 < y_2 \leq y_0$ (the analysis is same for the case of $y_2 < y_1 \leq y_0$). For $y \in (y_1, y_2]$,

$$g(y) = \alpha p_1 d(p_1) + (1 - \alpha) p_2 q_2 y - p q y$$

$$= [(\alpha p_1 q_1 + (1 - \alpha) p_2 q_2) - pq] y - \alpha p_1 [q_1 y - d(p_1)] < 0$$

(due to (14) and $q_1 y > d(p_1)$), and for $y \in (y_0, y_2]$, and for $y \in (y_2, y_0]$,

$$g(y) = \alpha p_1 d(p_1) + (1 - \alpha) p_2 d(p_2) - p q y$$

$$= [\ (\alpha p_1 q_1 + (1 - \alpha) p_2 q_2) - pq] y - \alpha p_1 [q_1 y - d(p_1)] - (1 - \alpha) p_2 [q_2 y - d(p_2)] < 0$$

(due to (14), $q_1 y > d(p_1)$, and $q_2 y > d(p_2)$). Thus, $g(y) \leq 0$ for all $y \in [y_L, y_H]$ (i.e., SS^- holds). In summary, SS^- holds under S1.

We now consider S2, for which $p_1 < p < p_2$,

$$d(p_1) > d(p) > d(p_2)$$

$$q_1 < q < q_2$$

which implies $d(p_2)/q_2 < d(p)/q < d(p_1)/q_1$, and $y_2 < y_0 < y_1$. For $y \geq y_1$, $g(y) \geq 0$ (see (13)). For $y \in [y_0, y_1)$,

$$g(y) = \alpha p_1 q_1 y + (1 - \alpha) p_2 d(p_2) - p d(p)$$

$$= \alpha p_1 d(p_1) + (1 - \alpha) p_2 d(p_2) - \alpha p_1 [d(p_1) - q_1 y] - p d(p) < 0$$

(due to (13) and $d(p_1) > q_1 y$). For $0 \leq y \leq y_2$,

$$g(y) = [\alpha p_1 q_1 + (1 - \alpha) p_2 q_2 - pq] y$$

$$= \{\alpha [p - (1 - \alpha)\delta_p][q - (1 - \alpha)\delta_q] + (1 - \alpha)[p + \alpha\delta_p][q + \alpha\delta_q]\} y - pqy$$

$$= \alpha (1 - \alpha) \delta_p \delta_q y \geq 0.$$

Thus, we need to show that there exists $y^* \in (y_2, y_0]$ such that

$$g(y) \geq 0 \text{ for } 0 \leq y \leq y^*,\ g(y^*) = 0, \text{ and } g(y) \leq 0 \text{ for } y > y^*. \tag{15}$$

For $y \in (y_2, y_0]$,

$$g'(y) = d[\alpha p_1 q_1 y + (1-\alpha)p_2 d(p_2) - pqy]/dy = \alpha p_1 q_1 - pq < 0 \qquad \text{(due to } p_1 < p \text{ and } q_1 < q\text{)}$$

$$g(y_0) = \alpha p_1 q_1 y_0 + (1-\alpha)p_2 d(p_2) - pd(p)$$

$$= \alpha p_1 d(p_1) + (1-\alpha)p_2 d(p_2) - \alpha p_1[d(p_1) - q_1 y_0] - pd(p) < 0 \qquad \text{(due to (13) } d(p_1) > q_1 y_0\text{)},$$

which ensures that there exists $y^* \in [y_2, y_0]$ such that (15) holds. Thus, SS^- holds in both scenarios.

Part (*b*): The proof follows the proof of part (*a*). We omit the details. ❒

Lemma 4. *For* P_D: *If* $r_p(p^*(q), z) \leq 0$ *for all* $z \leq z(p^*(q), q)$ *or if* $r_{pz}(p^*(q), z) \geq 0$ *for all* $z \leq z(p^*(q), q)$, *then* $p^o(q) < p^*(q)$.

Proof. Note that

$$D_p(p^*(q), q) = \int_{z_L}^{z(p^*(q),q)} U'\big(r(p^*(q),t) - cq\big) r_p(p^*(q),t)\phi_Z(t)\,dt + \int_{z(p^*(q),q)}^{z_H} U'\big((p^*(q)-c)q\big) q\phi_Z(t)\,dt$$

$$= U'\big((p^*(q)-c)q\big)\left[\int_{z_L}^{z(p^*(q),q)} h(t) r_p(p^*(q),t)\phi_Z(t)\,dt + \int_{z(p^*(q),q)}^{z_H} q\phi_Z(t)\,dt\right]$$

where $h(t) = \dfrac{U'\big(r(p^*(q),t) - cq\big)}{U'\big((p^*(q)-c)q\big)}$. Note that $h(t) > 1$ and $h'(t) < 0$ for $t < z(p^*(q), q)$, and $h(z(p^*(q), q)) = 1$. Therefore, from either $r_p(p^*(q), t) \leq 0$ or $r_{pz}(p^*(q), t) \geq 0$ for $t \leq z(p^*(q), q)$ and from

$$\bar{D}_p(p^*(q), q) = \int_{z_L}^{z(p^*(q),q)} r_p(p^*(q),t)\phi_Z(t)\,dt + \int_{z(p^*(q),q)}^{z_H} q\phi_Z(t)\,dt = 0,$$

it follows that $D_p(p^*(q), q) < \bar{D}_p(p^*(q), q) = 0$, which implies $p^o(q) < p^*(q)$. ❒

Proof of Proposition 1. Part (a) follows from Proposition 4. Part (*b*) follows from Lemma 2 and Lemma 3b. For part (c), Kocabıyıkoğlu and Popescu (2011) show that (3) is a sufficient condition for concave $\bar{D}(p,q)$ (see Proposition 2), and thus it follows from Proposition 1b that (3) is also a sufficient condition for concave $D(p, q)$. If the newsvendor is risk-neutral, then $u(p, q)/u^+(p, q) = 1$, and the concavity of $E\left[\tilde{S}(p,q,Y)\right]$ follows from Proposition 1a. Finally, Example 4 in Section A.1 of this appendix shows that (3) is not sufficient for concave $S(p, q)$. ❒

Proof of Proposition 2. The proof of Proposition 4 shows that expected profit and expected utility functions are strictly concave in price and in quantity (keeping the other variable fixed). Thus, unique univariate optimal solutions are obtained from the first-order condition, and this fact allows us to analyze the signs of the partial derivatives in the following proofs.

Part (*a*): Note that

$$S_q(p, q^*(p)) = \int_{y_L}^{d(p)/q^*(p)} U'\left(\tilde{S}\left(p, q^*(p), t\right)\right)\left(pt - c + s(t-1)\right)\phi_Y(t)\,dt -$$

$$\int_{d(p)/q^*(p)}^{y_H} U'\left(\tilde{S}\left(p, q^*(p), t\right)\right)\left(c - s(t-1)\right)\phi_Y(t)\,dt .$$

Let

$$u^+\left(p, q^*(p)\right) = \int_{y_L}^{d(p)/q^*(p)} U'\left(\tilde{S}\left(p, q^*(p), t\right)\right)\left(\frac{\left(pt - c + s(t-1)\right)\phi_Y(t)}{\int_{y_L}^{d(p)/q^*(p)} \left(pt - c + s(z-1)\right)\phi_Y(z)\,dz}\right)dt$$

$$u^-\left(p, q^*(p)\right) = \int_{d(p)/q^*(p)}^{y_H} U'\left(\tilde{S}\left(p, q^*(p), t\right)\right)\left(\frac{\left(c - s(t-1)\right)\phi_Y(t)}{\int_{d(p)/q^*(p)}^{y_H} \left(c - s(z-1)\right)\phi_Y(z)\,dz}\right)dt .$$

Note that

$py - c + s(y-1) > 0$ for all $y \in (y_L, y_H]$ (due to $y_L \geq \frac{c+s}{p+s}$) (16)

$-c + s(y-1) < 0$ for all $y \in [y_L, y_H)$ (due to $y_H \leq \frac{c+s}{s}$) (17)

$d(p)/q^*(p) \in (y_L, y_H)$ (due to $\bar{S}_q\left(p, d(p)/y_H\right) > 0,\ \bar{S}_q\left(p, d(p)/y_L\right) < 0$) (18)

$\int_{y_L}^{d(p)/q^*(p)} \left(pt - c + s(t-1)\right)\phi_Y(t)\,dt > 0$ (due to (16) and (18)) (19)

$u^+\left(p, q^*(p)\right) > u^-\left(p, q^*(p)\right) > 0$ (due to $U'' < 0$, (16) – (18))

Therefore

$$S_q(p, q^*(p)) = u^+\left(p, q^*(p)\right) \int_{y_L}^{d(p)/q^*(p)} \left(pt - c + s(t-1)\right)\phi_Y(t)\,dt -$$

$$u^{-}\left(p,q^{*}(p)\right)\int_{d(p)/q^{*}(p)}^{y_H}\left(c-s(t-1)\right)\phi_Y(t)\,dt$$

$$>u^{-}\left(p,q^{*}(p)\right)\left[\int_{y_L}^{d(p)/q^{*}(p)}\left(pt-c+s(t-1)\right)\phi_Y(t)\,dt-\int_{d(p)/q^{*}(p)}^{y_H}\left(c-s(t-1)\right)\phi_Y(t)\,dt\right]$$

$$=u^{-}\left(p,q^{*}(p)\right)\overline{S}_q\left(p,q^{*}(p)\right)=0,$$

which implies $q^{o}(p) > q^{*}(p)$.

Part (*b*):

$$S_p(p^{*}(q),q)=\int_{y_L}^{d(p^{*}(q))/q}U'\left(p^{*}(q)qt-cq\right)qt\phi_Y(t)\,dt+$$

$$\int_{d(p^{*}(q))/q}^{y_H}U'\left(p^{*}(q)d\left(p^{*}(q)\right)-cq\right)r_p\left(p^{*}(q)\right)\phi_Y(t)\,dt$$

Note that

$r_p(p^{*}(q)) < 0$ (due to $\overline{S}_p\left(p^{*}(q),q\right)=0$)

$U'\left(p^{*}(q)qy-cq\right)>U'\left(p^{*}(q)d\left(p^{*}(q)\right)-cq\right)$ for all $y < d(p^{*}(q))/q$ (due to $U'' < 0$).

Therefore

$$S_p(p^{*},q)>U'\left(p^{*}(q)d\left(p^{*}(q)\right)-cq\right)\left[\int_{y_L}^{d(p^{*}(q))/q}qt\phi_Y(t)\,dt+\int_{d(p^{*}(q))/q}^{y_H}r_p\left(p^{*}(q)\right)\phi_Y(t)\,dt\right]$$

$$=U'\left(p^{*}(q)d\left(p^{*}(q)\right)-cq\right)\overline{S}_p\left(p^{*}(q),q\right)=0,$$

which implies $p^{o}(q) > p^{*}(q)$.

Part (*c*): This result appears in Eeckhoudt et al. (1995).

Part (*d*): If $d(p, z) = d(p)z$, then the support of Z must be nonnegative (i.e., to avoid the impractical possibility of negative demand). Let p^1 denote the price that maximizes expected revenue $E[pd(p)Z]$, i.e.,

$$\int_{z_L}^{z_H}\left[d\left(p^1\right)+p^1d'\left(p^1\right)\right]t\phi_Z(t)\,dt=0\,.$$

From $z_L \geq 0$, it follows that $d(p^1) + p^1d'(p^1) = 0$. Therefore,

$$\overline{D}_p\left(p^1,q\right)=\int_{z_L}^{z(p^1,q)}\left[d\left(p^1\right)+p^1d'\left(p^1\right)\right]t\phi_Z(t)\,dt+\int_{z(p^1,q)}^{z_H}q\phi_Z(t)\,dt>0\,.$$

Thus, $p^{*}(q) > p^1$, which implies

$$d\left(p^*(q)\right)+p^*d'\left(p^*\right)\le d\left(p^1\right)+p^1d'\left(p^1\right)=0$$

(due to $r_{pp} < 0$). Therefore $r_p\left(p^*(q),z\right)=\left[d\left(p^*(q)\right)+p^*(q)d'\left(p^*(q)\right)\right]z\le 0$ and the result holds by Lemma 4.

Part (*e*): If $d(p, z) = d(p) + z$, then $r_{pz}(p, z) = 1$, and the result holds by Lemma 4. ❐

Proof of Proposition 3. Before considering parts (a) and (b), we develop a result regarding the nature of the $q^*(p)$ and $p^*(q)$ curves that applies to both P_S and P_D. For this purpose, we generically represent the expected profit function for either problem as $F(p, q)$. The proof of Proposition 4 shows that expected profit functions for P_S and P_D are strictly concave in price and in quantity (keeping the other variable fixed), i.e., $F_{pp} < 0$ and $F_{qq} < 0$. By the implicit function theorem,

$$p^{*\prime}(q)=\frac{-F_{pq}\left(p^*(q),q\right)}{F_{pp}\left(p^*(q),q\right)}$$

$$q^{*\prime}(p)=\frac{-F_{pq}\left(p,q^*(p)\right)}{F_{qq}\left(p,q^*(p)\right)}.$$

From the fact that F has a unique stationary point (p^*, q^*) that is a global maximum, it follows that

$$p^{*\prime}(q^*)q^{*\prime}(p^*)=\frac{F_{pq}\left(p^*,q^*\right)^2}{F_{pp}\left(p^*,q^*\right)F_{qq}\left(p^*,q^*\right)}<1 \tag{20}$$

(i.e., the determinant of the Hessian of F is positive). We relate $p^{*\prime}(q)$ to $p^{*-1\prime}(p)$ via implicit differentiation of $p^*(p^{*-1}(p)) = p$, i.e.,

$$0=\frac{d}{dp}[0]=\frac{d}{dp}\left[p^*\left(p^{*-1}(p)\right)-p\right]=p^{*\prime}\left(p^{*-1}(p)\right)p^{*-1\prime}(p)-1,$$

which can be rewritten as

$$p^{*\prime}(q) = 1/p^{*-1\prime}(p). \tag{21}$$

By supposition, $q^{*\prime}(p) \ge 0$, $p^{*\prime}(q) \ge 0$, and the curves $p^{*-1}(p)$ and $q^*(p)$ intersect at a single point that is the optimum. Therefore, from (20) and (21) it follows that $p^{*-1\prime}(p^*) > q^{*\prime}(p^*) \ge 0$, i.e., $p^{*-1\prime}(p)$ intersects $q^*(p)$ once from below, as illustrated in Figure 1.

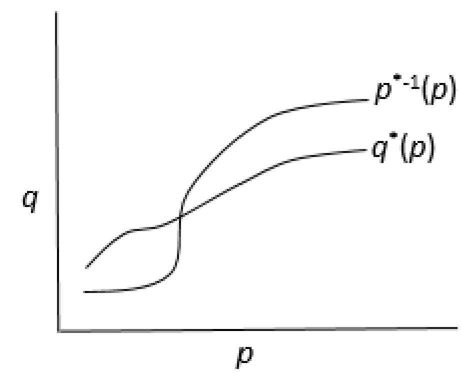

Figure 1. Illustration of $p^{*-1\prime}(p)$ and $q^*(p)$. Both curves have nonnegative slope with $p^{*-1\prime}(p)$ intersecting $q^*(p)$ once and from below.

Part (a): From the fact that $S(p, q)$ has a unique stationary point that is a global maximum, it follows that curves $q^o(p)$ and $p^o(q)$ intersect at a single point that is a global maximum. From propositions 2a and 2b, $q^o(p) > q^*(p)$ for all $p \geq p^*$, and $p^o(q) > p^*(q)$. Thus, curve $q^o(p)$ is an upward shift of curve $q^*(p)$ for $p \geq p^*$ and curve $p^o(q)$ is an upward shift of $p^*(q)$ (equivalently, a right-shift of curve $p^{*-1\prime}(p)$), and it follows that the intersection point (p^o, q^o) must be up and to the right of (p^*, q^*), i.e., $p^o > p^*$ and $q^o > q^*$.

Part (b): From the fact that $D(p, q)$ has a unique stationary point that is a global maximum, it follows that curves $q^o(p)$ and $p^o(q)$ intersect at a single point that is a global maximum. From propositions 2c – 2e, $q^o(p) < q^*(p)$ and $p^o(q) < p^*(q)$. Thus, curve $q^o(p)$ is a downward shift of curve $q^*(p)$ and curve $p^o(q)$ is a downward shift of $p^*(q)$ (equivalently, a left-shift of curve $p^{*-1\prime}(p)$), and it follows that the intersection point (p^o, q^o) must be down and to the left of (p^*, q^*), i.e., $p^o < p^*$ and $q^o < q^*$. ❐