Online Appendix

Technical Note – Pricing Below Cost under Exchange-Rate Risk

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Proof of Proposition 1. For any price level, the second-order derivative of the objective function (1) with respect to X is zero, implying that the objective function is linearly increasing in the production amount. For this reason, it is sufficient to consider the first-order derivatives to complete the proof. We provide the details for the case of $0 \le X \le \min\{d_H, d_F\}$. Similar logic can be applied to obtain the results for the remaining three cases.

i) When $0 \le X \le \min\{d_H, d_F\}$, $E\left[\Pi\left(p_H, p_F, X\right)\right] = -cX + \int_{e_l}^{\bar{e}} \left(p_H - t_H\right) Xf\left(e\right) de + \int_{\bar{e}}^{e_h} \left(p_F - t_F\right) eXf\left(e\right) de$

and $\partial E\left[\Pi\left(p_H, p_F, X\right)\right] / \partial X = -c + \left(p_H - t_H\right)\left(1+\theta\right)$. This implies that if $p_H < c/(1+\theta) + t_H$, the optimal production amount is $X^* = 0$; otherwise, if $p_H \ge c/(1+\theta) + t_H$, then the optimal production amount is at least $\min\{d_H, d_F\}$. ❑

Proof of Proposition 2. We again take advantage of the fact that the three decision variables (p_H, p_F, X) in (1) reduces to two decision variables (p_H, X) because $p_F = p_H - t_H + t_F$.

Part (a): From the first- and second-order condition of (5), we derive the following, respectively:
$\partial E\left[\Pi^{TD}\right] / \partial p_H = d_H + \left[\left(p_H - t_H\right) - c\right] d_H' + d_F + \left[\left(p_H - t_H\right) - c\right] d_F'$, and
$\partial^2 E\left[\Pi^{TD}\right] / \partial p_H^{\,2} = \left[2d_H' + \left(p_H - t_H - c\right) d_H''\right] + \left[2d_F' + \left(p_H - t_H - c\right) d_F''\right] \le 0$, and thus policy TD is concave in p_H. The same logic can be applied to show that the expected profit functions under policies PHX and PHN are concave.

Part (b): The proof follows from the first-order condition of the expected profit function where p_H is small such that $0 \le X \le d_m$:

$$\partial E\left[\Pi\left(p_H, X \middle| 0 \le X \le d_m\right)\right] / \partial p_H = \int_{e_l}^{\bar{e}} [X] f(e) de + \int_{\bar{e}}^{e_h} [eX] f(e) de = (1+\theta) X > 0$$

$$\partial^2 E\left[\Pi\left(p_H, X \middle| 0 \le X \le d_m\right)\right] / \partial p_H{}^2 = 0 .$$

The first derivative shows that the objective function is always increasing in p_H for a given X in this interval, and because the expected profit is linearly increasing in p_H for a given X in this interval, the optimal price is always at least equal to the amount that would satisfy the minimum demand. From Proposition 1, we know that the minimum demand is produced when price is greater than equal to $[c/(1+\theta)] + t_H$. Thus, the optimal solution would always settle at the quantity resulting in the minimum demand and the price at $p \geq [c/(1+\theta)] + t_H$. Because PHN returns zero expected profit at $p = [c/(1+\theta)] + t_H$ and $X = d_m$, policy NP can be eliminated from the potentially optimal solutions.

Part (c): Proposition 2a has already established that policies TD, PHX and PHN are concave in p_H. Thus, it is sufficient to consider the first-order derivatives with respect to p_H to complete the proof.

PHN policy: From the first-order condition of (7), we have: $\partial E\left[\Pi^{PHN}\right] / \partial p_H = d_m(1+\theta) + (p_H - t_H - c) d_m' + (p_H - t_H) d_m' \theta = 0$. This provides the optimal price $p^{PHN} = [c/(1 + \theta)] + t_H + [d_m/(-d_m')]$. By rearranging the terms of p^{PHN}, we have $X^{PHN} = [p^{PHN} - t_H - (c/(1 + \theta))][-d_m']$. And, by definition, $X^{PHN} = d_m$, and thus from Proposition 1, we have $p^{PHN} < c$. Similar logic can be applied to obtain the results for the PHX and TD policies. ❑

Necessary and Sufficient Condition for PHN policy to be optimal:

Recall the objective functions of policies TD, PHX and PHN in equations (5)–(7). Substituting the optimal price expressions that maximize (5)–(7) (also in Table 1), we get the following expected profit expressions:

$$E[\Pi^{TD}(p_H{}^{TD})] = (d_x{}^{TD} + d_m{}^{TD})^2/[-(d_x{}^{TD\prime} + d_m{}^{TD\prime})] \quad (18)$$

$$E[\Pi^{PHX}(p_H{}^{PHX})] = [(d_x{}^{PHX} + \theta d_m{}^{PHX})^2 - cd_x{}^{PHX\prime}(d_x{}^{PHX} + \theta d_m{}^{PHX})]/[-(d_x{}^{PHX\prime} + \theta d_m{}^{PHX\prime})] \quad (19)$$

$$E[\Pi^{PHN}(p_H{}^{PHN})] = (1 + \theta)\,(d_m{}^{PHN})^2/[-(d_m{}^{PHN\prime})] \quad (20)$$

Thus, PHN policy is optimal when $(1 + \theta)\,(d_m{}^{PHN})^2/[-(d_m{}^{PHN\prime})] > \max\{(d_x{}^{TD} + d_m{}^{TD})^2/[-(d_x{}^{TD\prime} + d_m{}^{TD\prime})], [(d_x{}^{PHX} + \theta d_m{}^{PHX})^2 - cd_x{}^{PHX\prime}(d_x{}^{PHX} + \theta d_m{}^{PHX})]/[-(d_x{}^{PHX\prime} + \theta d_m{}^{PHX\prime})]\}$.

Expected profit expressions in (18) – (20) can be rewritten as:

$$E[\Pi^{TD}(p_H{}^{TD})] = p_H{}^{TD}(d_x{}^{TD}(p_H{}^{TD}) + p_H{}^{TD} d_m{}^{TD}(p_H{}^{TD}))/\varepsilon^{TD}(p_H{}^{TD}) \quad (21)$$

$$E[\Pi^{PHX}(p_H{}^{PHX})] = p_H{}^{PHX}(d_x{}^{PHX}(p_H{}^{PHX}) + \theta d_m{}^{PHX}(p_H{}^{PHX}))/\varepsilon^{PHX}(p_H{}^{PHX}) - c\{\theta[d_x(p_H{}^{PHX})d_m'(p_H{}^{PHX}) - d_x'(p_H{}^{PHX})d_m(p_H{}^{PHX})]/[d_x'(p_H{}^{PHX}) + \theta\, d_m'(p_H{}^{PHX})]\} \quad (22)$$

$$E[\Pi^{PHN}(p_H{}^{PHN})] = (1 + \theta) p_H{}^{PHN} d_m(p_H{}^{PHN})/\varepsilon_N(p_H{}^{PHN}) \quad (23)$$

For the TD policy in (21), the optimal price solves

$$(p_H{}^{TD} - c)/p_H{}^{TD} = 1/\,\varepsilon^{TD}(p_H{}^{TD}). \quad (24)$$

For the PHX policy in (22), the optimal price solves

$$(p_H^{PHX} - c(d_x'(p_H^{PHX})/(d_x^{PHX}(p_H^{PHX}) + \theta d_m^{PHX}(p_H^{PHX}))))/p_H^{PHX} = 1/\,\varepsilon^{PHX}(p_H^{PHX}). \tag{25}$$

For the PHN policy in (23), the optimal price solves

$$(p_H^{PHN} - c/(1+\theta))/p_H^{PHN} = 1/\,\varepsilon^{PHN}(p_H^{PHN}). \tag{26}$$

Lemma A1. *If $p_H^{PHX} < c + t_H$, then $p_H^{PHN} < c + t_H$ and $E[\Pi^{PHN}(p_H^{PHN})] > E[\Pi^{PHX}(p_H^{PHX})]$.*

Proof. If $p_H^{PHX} < c + t_H$, then

$$\begin{aligned} E[\Pi^{PHX}(p_H^{PHX})] &= (p_H^{PHX} - c - t_H)d_x(p_H^{PHX}) + \theta(p_H^{PHX} - t_H)d_m(p_H^{PHX}) \\ &= E[\Pi^{PHN}(p_H^{PHX})] - (c + t_H - p_H^{PHX})[d_x(p_H^{PHX}) - d_m(p_H^{PHX})] < E[\Pi^{PHN}(p_H^{PHN}) \end{aligned} \tag{27}$$

If $p_H^{PHN} \geq c + t_H$, then

$$\begin{aligned} E[\Pi^{PHN}(p_H^{PHN})] &= (p_H^{PHN} - c - t_H)d_m(p_H^{PHN}) + \theta(p_H^{PHN} - t_H)d_m(p_H^{PHN}) \\ &= E[\Pi^{PHX}(p^{PHN})] - (p^{PHN} - c - t_H)[d_x(p^{PHN}) - d_m(p^{PHN})] \leq E[\Pi^{PHX}(p^{PHX})], \end{aligned}$$

which contradicts (27). Therefore $p_H^{PHN} < c + t_H$ and $E[\Pi^{PHN}(p_H^{PHN})] > E[\Pi^{PHX}(p_H^{PHX})]$. ❑

Lemma A2. *The following inequalities on optimal prices are not possible simultaneously: $p_H^{TD} < c/(1-\theta) + t_H$ and $p^{PHX} > c/(1-\theta) + t_H$.*

Proof. If $p^{TD} < c/(1-\theta) + t_H$, then from Proposition 1 we have $E[\Pi^{TD}(p_H^{TD})] < E[\Pi^{PHX}(p_H^{TD})]$. If $p_H^{PHX} \geq c/(1-\theta) + t_H$, then $E[\Pi^{PHX}(p_H^{PHX})] \leq E[\Pi^{TD}(p_H^{PHX})]$. But by definition $E[\Pi^{PHX}(p_H^{TD})] < E[\Pi^{PHX}(p_H^{PHX})]$ and $E[\Pi^{TD}(p_H^{PHX})] < E[\Pi^{TD}(p_H^{TD})]$, which implies $E[\Pi^{TD}(p_H^{TD})] < E[\Pi^{PHX}(p_H^{TD})] < E[\Pi^{PHX}(p_H^{PHX})]$ and $E[\Pi^{PHX}(p_H^{PHX})] < E[\Pi^{TD}(p_H^{PHX})] \leq E[\Pi^{TD}(p_H^{TD})]$, which is a contradiction. ❑

Proof of Proposition 3. PHN is uniquely optimal when (1) $p_H^{PHN} < c + t_H$, (2a) $p_H^{PHX} < c + t_H$ or (2b) $p_H^{PHX} > [c/(1-\theta)] + t_H$, and (3) $p^{TD} < [c/(1-\theta)] + t_H$. Lemma A2 shows that conditions (2b) and (3) cannot occur simultaneously, and therefore, PHN is uniquely optimal when conditions (1), (2a), and (3) occur simultaneously. Lemma A1 shows that when condition (2a) occurs, condition (1) automatically holds, and therefore, the sufficient condition is reduced to satisfying conditions in (2a) and (3) simultaneously. Substituting the condition in (3) into (24) at the price point of $[c/(1-\theta)] + t_H$ provides: $\varepsilon^{TD}((c/(1-\theta)) + t_H) > (c + t_H(1-\theta))/(c\theta + t_H(1-\theta))$. Similarly, substituting the condition in (2a) at the price point of $c + t_H$ into (25) provides: $\varepsilon^{PHX}(c + t_H) > 1 + [d_x'(c + t_H)/(\theta d_x'(c + t_H))]$. Thus, the PHN policy is optimal and the optimal price is below cost when $\varepsilon^{TD}((c/(1-\theta)) + t_H) > (c + t_H(1-\theta))/(c\theta + t_H(1-\theta))$ and $\varepsilon^{PHX}(c + t_H) > 1 + [d_x'(c + t_H)/(\theta d_x'(c + t_H))]$. ❑

Proof of Proposition 4. Part (a): The first-order derivative of the absolute value of the risk premium in (10) w.r.t. θ provides the proof: $\partial\left|r_H^{PHN}\right|/\partial\theta = c\left[1/(1+\theta)^2\right] > 0$. Part (b): The first-order derivative of the PHN price in Table 1 w.r.t. θ provides the proof: $\partial p_H^{PHN}/\partial\theta = -\,\mathrm{c}\,[1/(1+\theta)^2] < 0$. Part (c): The first-order derivative of the expected profit function for the PHN policy as expressed in (7) provides the proof: $\partial E[\Pi^{PHN}(X = d_H \leq d_F)]/\partial\theta = pd_H > 0$ and $\partial E[\Pi^{PHN}(X = d_F \leq d_H)]/\,\partial\theta = pd_F > 0$. ❑

Proof of Proposition 5. Part (a): The first-order derivative of the PHN price in Table 1 w.r.t. c provides the proof: $\partial p_H^{PHN}/\partial c = 1/(1+\theta) > 0$. Part (b): The first-order derivative of the absolute value of the risk premium in (10) w.r.t. c provides the proof: $\partial r_H^{PHN}/\partial c = \theta/(1+\theta) > 0$. Part (c): Because $\theta < 1$, $\partial p_H^{PHN}/\partial c - \partial r_H^{PHN}/\partial c = (1-\theta)/(1+\theta) > 0$. ❑

Proposition A1. *a) The objective function* (1) *is piecewise linear in X for a given p_H. b) For a given price level p_H, there are six potentially optimal production policies:*

i) $[c/(1+\theta)] + t_H \le p_H < \min\{\max\{c/(1+\theta) + t_H, c - (\beta/d_m) + t_H\}, c + t_H\}$, *then* $X^* = \beta/(c - p_H + t_H) < d_m$ $\Rightarrow$ PHIN

ii) $\min\{\max\{[c/(1+\theta)] + t_H, c - (\beta/d_m) + t_H\}, c + t_H\} \le p_H < c + t_H$, *then* $X^* = d_m \Rightarrow$ PHN

iii) $c \le p_H < \min\{\max\{c + t_H, [(cd_F - \beta)/(d_H(1-e_\alpha)+d_F e_\alpha)] + t_H\}, [c/(1-\theta)] + t_H\}$, *then* $d_H < X^* = ((1-e_\alpha)d_H + (\beta/(p_H+t_H)))/((c/(p_H+t_H)) - e_\alpha) < d_F \Rightarrow$ PHIX

iv) $\min\{\max\{c + t_H, [(cd_F - \beta)/(d_H(1-e_\alpha)+d_F e_\alpha)] + t_H\}, [c/(1-\theta)] + t_H\} \le p_H \le [c/(1-\theta)] + t_H$, *then* $X^* = d_x \Rightarrow$ PHX

v) $[c/(1-\theta)] + t_H < p_H < \max\{[((d_H+d_F)c - \beta)/(d_H + d_F e_\alpha)] + t_H, [c/(1-\theta)] + t_H\}$, *then* $d_x < X^* = ((1-e_\alpha)d_H + (\beta/(p_H - t_H)))/((c/(p_H - t_H)) - e_\alpha) < d_H + d_F \Rightarrow$ PHI

vi) $\max\{[((d_H + d_F)c - \beta)/(d_H + d_F e_\alpha)] + t_H, [c/(1-\theta)] + t_H\} \le p_H$, *then* $X^* = d_H + d_F \Rightarrow$ TD

c) Policy PHX *satisfies the risk constraint* (11) *and dominates policy* PHIX *when $d_H \ge d_F$ for $c + t_H \le p_H < [c/(1-\theta)] + t_H$.*

Proof of Proposition A1. Part (a): For any price p_H, the second-order derivative of the objective function with respect to X is zero, implying that the objective function is linearly increasing in the production amount. For this reason, it is sufficient to consider the first-order derivatives to complete the proof.

Parts (bi) and (bii): When $0 \le X \le d_m$,

$$E\left[\Pi\left(p_H, X\right)\right] = -cX + \int_{e_l}^{\bar{e}} \left(p_H - t_H\right) X f\left(e\right) de + \int_{\bar{e}}^{e_h} \left(p_H - t_H\right) eXf\left(e\right) de\,;$$

$\partial E\left[\Pi\left(p_H, X\right)\right]/\partial X = -c + \left(p_H - t_H\right)\left(1+\theta\right)$. This implies that if $p < [c/(1+\theta)] + t_H$, the optimal production amount is $X^* = 0$; otherwise, if $[c/(1+\theta)] + t_H$, then the optimal production amount is at least d_m if (11) is satisfied. When $d_m \le X \le d_x$, if the home demand is larger than the foreign demand at a price level p_H, then the objective function can be written as follows:

$$E\left[\Pi\left(p_H, d_F < X < d_H\right)\right] = -cX + \int_{e_l}^{\bar{e}} \left(p_H - t_H\right) Xf\left(e\right) de + \int_{\bar{e}}^{e_h} \left[\left(p_H - t_H\right) ed_F + \left(p_H - t_H\right)\left(X - d_F\right)\right] f\left(e\right) de,$$

$\partial E\left[\Pi\left(p, X\right)\right]/\partial X = -c + p_H - t_H$ or, alternatively if the foreign demand is higher than the domestic demand at p_H, then,

$$E\left[\Pi\left(p_H, d_H < X < d_F\right)\right] = -cX + \int_{e_l}^{\bar{e}}\left[\left(p_H - t_H\right)d_H + \left(p_H - t_H\right)e\left(X - d_H\right)\right]f\left(e\right)de + \int_{\bar{e}}^{e_h}\left(p_H - t_H\right)eXf\left(e\right)de,$$

$\partial E\left[\Pi\left(p_H, X\right)\right]/\partial X = -c + p_H + t_H$. Both cases imply that if $p < c + t_H$, the optimal production amount is $X^* = d_m$ if (11) is satisfied; otherwise, if $p \geq c + t_H$, then the optimal production amount is at least d_x if (11) is satisfied. However, if $p < c + t_H - (\beta/d_m)$ then (11) is violated. Thus, if $[c/(1+\theta)] + t_H \leq p < c + t_H - (\beta/d_m)$, then it is optimal to produce the maximum amount while satisfying (11). Because (11) is binding, we have $X^* = \beta/(c - p + t_H)$ which we label as the PHIN policy. If $\min\{\max\{[c/(1+\theta)] + t_H, c - (\beta/d_m) + t_H\}, c + t_H) \leq p_H < c + t_H$, then $X^* = d_m$ since (11) is satisfied which we label as the PHN policy.

Parts (biii) and (biv): Consider the case when $d_x \leq X \leq d_H + d_F$.

$$E\left[\Pi\left(p_H, d_x \leq X < d_H + d_F\right)\right] = -cX + \left\{\begin{array}{l}\int_{e_l}^{\bar{e}}\left[\left(p_H - t_H\right)d_H + \left(p_H - t_H\right)e\left(X - d_H\right)\right]f\left(e\right)de \\ + \int_{\bar{e}}^{e_h}\left[\left(p_H - t_H\right)ed_F + \left(p_H - t_H\right)\left(X - d_F\right)\right]f\left(e\right)de\end{array}\right\},$$

$\partial E\left[\Pi\left(p_H, d_x X < d_H + d_F\right)\right]/\partial X = -c + \left(p_H - t_H\right)\left(1 - \theta\right)$. This implies that if $p_H < [c/(1 - \theta)] + t_H$, then the optimal production amount is $X^* = d_x$ if (11) is satisfied; otherwise, if $p_H \geq [c/(1 - \theta)] + t_H$, then the optimal production amount is $d_H + d_F$ if (11) is satisfied. If $d_H \geq d_F$, $X^* = d_x$ always satisfies (11) for $c + t_H \leq p_H < [c/(1 - \theta)] + t_H$ which we label as the PHX policy. However for the same price region, if $d_H < d_F$ and $p_H < [(cd_F - \beta)/(d_H(1 - e_\alpha) + d_F e_\alpha)] + t_H$, then $X^* = d_x$ violates (11). Thus, if $d_H < d_F$ for $c + t_H \leq p_H < \min\{\max\{c + t_H, [(cd_F - \beta)/(d_H(1 - e_\alpha) + d_F e_\alpha)] + t_H\}, [c/(1 - \theta)] + t_H\}\}$, then it is optimal to produce the maximum amount while satisfying (11). Because (11) is binding, we have $X^* = ((1 - e_\alpha)d_H + (\beta/(p_H - t_H)))/((c/(p_H - t_H)) - e_\alpha)$ which we label as the PHIX policy. If $d_H < d_F$ for $\min\{\max\{c + t_H, [(cd_F - \beta)/(d_H(1 - e_\alpha) + d_F e_\alpha)] + t_H\}\}, [c/(1 - \theta))] + t_H \leq p_H \leq [c/(1 - \theta)] + t_H$, then $X^* = d_x$ since (11) is satisfied, which we label as the PHX policy.

Parts (b-v) and (b-vi): If $p_H \geq [c/(1 - \theta)] + t_H$, then the optimal production amount is $X^* = d_H + d_F$ if (11) is satisfied. However, if $p_H < [((d_H + d_F)c - \beta)/(d_H + d_F e_\alpha)] + t_H$, then (11) is violated. Thus, if $[c/(1 - \theta)] + t_H < p_H < [((d_H + d_F)c - \beta)/(d_H + d_F e_\alpha)] + t_H$ then it is optimal to produce the maximum amount while satisfying (11). Because (11) is binding, we have $X^* = ((1 - e_\alpha)d_H + (\beta/(p_H - t_H)))/((c/(p_H - t_H)) - e_\alpha)$ which we label as the PHI policy. If $\max\{[c/(1 - \theta)] + t_H, [((d_H + d_F)c - \beta)/(d_H + d_F e_\alpha)] + t_H\} \leq p_H$, then $X^* = d_H + d_F$ since (11) is satisfied which we label as the TD policy. Thus for a given price level, there are six potentially optimal production policies.

Part (c): For $c + t_H \leq p_H < [c/(1 - \theta)] + t_H$, it is already proven that it is optimal to produce $X^* = d_x$. When $d_H > d_F$ in this case, the PHX policy always satisfies (11) because the firm can sell all of its products in the home market at low exchange rate realizations without a chance for a loss. ❑

Proof of Proposition 6. Part (a): For any price level p_H, the second-order derivative of the objective function with respect to X is zero, implying that the objective function is linearly increasing in the production amount. For this reason, it is sufficient to consider the first-order derivatives to complete the proof.

Parts *i*) and *ii*): When $0 \le X \le d_m$, $E\left[\Pi\left(p_H, X\right)\right] = -cX + \int_{e_l}^{\bar{e}} \left(p_H - t_H\right) Xf\left(e\right) de + \int_{\bar{e}}^{e_h} \left(p_H - t_H\right) eXf\left(e\right) de$;

$\partial E\left[\Pi\left(p_H, X\right)\right] / \partial X = -c + \left(p_H - t_H\right)\left(1+\theta\right)$. This implies that if $p_H < [c/(1+\theta)] + t_H$, the optimal production amount is $X^* = 0$; otherwise, if $p_H \ge [c/(1+\theta)] + t_H$, then the optimal production amount is at least d_m if (11) is satisfied. When $d_m \le X \le d_x$, if the home demand is larger than the foreign demand at a price level p_H, then the objective function can be written as follows:

$$E\left[\Pi\left(p_H, d_F < X < d_H\right)\right] = -cX + \int_{e_l}^{\bar{e}} \left(p_H - t_H\right) Xf\left(e\right) de + \int_{\bar{e}}^{e_h} \left[\left(p_H - t_H\right) ed_F + \left(p_H - t_H\right)\left(X - d_F\right)\right] f\left(e\right) de,$$

$\partial E\left[\Pi\left(p_H, X\right)\right] / \partial X = -c + p - t_H$ or, alternatively if the foreign demand is higher than the domestic demand at p_H, then,

$$E\left[\Pi\left(p_H, d_H < X < d_F\right)\right] = -cX + \int_{e_l}^{\bar{e}} \left[\left(p_H - t_H\right) d_H + \left(p_H - t_H\right) e\left(X - d_H\right)\right] f\left(e\right) de + \int_{\bar{e}}^{e_h} \left(p_H - t_H\right) eXf\left(e\right) de,$$

$\partial E\left[\Pi\left(p_H, X\right)\right] / \partial X = -c + p_H - t_H$. Both cases imply that if $p_H < c + t_H$, the optimal production amount is $X^* = d_m$ if (11) is satisfied. However, if $p_H < c - (\beta/d_m) + t_H$ then (11) is violated. Thus, if $[c/(1+\theta)] + t_H \le p_H < c - (\beta/d_m) + t_H$, then it is optimal to produce the maximum amount while satisfying (11). Because (11) is binding, we have $X^* = \beta/(c - p_H + t_H)$ which we label as the PHIN policy. If $\min\{\max\{[c/(1+\theta)] + t_H, c - (\beta/d_m) + t_H\}, c + t_H\} \le p_H < c + t_H$, then $X^* = d_m$ since (11) is satisfied which we label as the PHN policy.

Part (b): Proposition 2 has already established that the PHN policy is concave in p_H. From the first-order condition of (7), we have: $\partial E\left[\Pi^{PHN}\right] / \partial p_H = d_m\left(1+\theta\right) + \left(p_H - c - t_H\right) d_m{}' + \theta\left(p_H - t_H\right) d_m{}' = 0$ provides the optimal price choice $p_H^{PHN} = [c/(1 + \theta)] + [d_m/(-d_m')] + t_H$. Part (a) has established that $X^* = \beta/(c - p_H + t_H)$ under the PHIN policy, which is still equivalent to the minimum demand level d_m. Equating $X^* = \beta/(c - p_H + t_H) = d_m$ and solving for p_H leads to the optimal price choice under the PHIN policy, i.e., $p_H^{PHIN} = c - (\beta/d_m) + t_H$. ❑

Proof of Proposition 7. Part (a): The optimal price under the PHIN policy is $p_H^{PHIN} = c - (\beta/d_m) + t_H$. The first-order derivative w.r.t. β provides the proof: $\partial p^{PHIN}/\partial\beta = -1/d_m < 0$. Thus, the value of p_H^{PHIN} increases with smaller values of β. Part (b): The risk discount can be calculated as

$r_H^{PHIN} = p_H^0 - p_H^{PHIN} = [c + [(d_H + d_F)/(-(d_H' + d_F'))] + t_H] - [c - (\beta/d_m) + t_H] = (\beta/d_m) + [(d_H + d_F)/(-(d_H' + d_F'))] > 0$ because both terms (β/d_m) and $[(d_H + d_F)/(-(d_H' + d_F'))]$ are positive. Moreover, $\partial r^{PHIN}/\partial\beta = (1/d_m) > 0$; thus, the value of the risk discount is decreasing with smaller values of β. ❑

Proposition A2. *When* $\max\{y_H, y_F\} < \max\{d_H, d_F\} - \min\{d_H, d_F\}$, *the optimal manufacturing quantity under the* PHN *policy is* $X^{PHN} = \min\{d_H, d_F\} + \max\{y_H, y_F\}$.

Proof of Proposition A2. We show the proof for the case when $d_H < d_F$ and the proof for the case when $d_F < d_H$ follows a similar approach. Expected profit for the PHN policies under $\max\{y_H, y_F\} < d_F - d_H$ are:

$$E\left[\Pi\left(p_H, X = d_H + y_H\right)\right] = -c\left(d_H + y_H\right) + \int_{e_l}^{\bar{e}} \begin{bmatrix} \left(p_H - t_H\right)\left(d_H + y_H - y_F\right) \\ +\left(p_H - t_H\right)ey_F \end{bmatrix} f\left(e\right)de$$
$$+\int_{\bar{e}}^{e_h}\left[\left(p_H - t_H\right)y_H + \left(p_H - t_H\right)ed_H\right]f\left(e\right)de$$
$$= \left[\left(p_H - t_H\right)\left(1+\theta\right) - c\right]d_H + \left(p_H - t_H - c\right)y_H - \left(p_H - t_H\right)y_H\theta$$

$$E\left[\Pi\left(p_H, X = d_H + y_F\right)\right] = -c\left(d_H + y_F\right) + \int_{e_l}^{\bar{e}}\left[\left(p_H - t_H\right)d_H + \left(p_H - t_H\right)ey_F\right]f\left(e\right)de$$
$$+\int_{\bar{e}}^{e_h}\left[\left(p_H - t_H\right)y_H + \left(p_H - t_H\right)e\left(d_H + y_F - y_H\right)\right]f\left(e\right)de$$
$$= \left[\left(p_H - t_H\right)\left(1+\theta\right) - c\right]d_H + \left(p_H - t_H - c\right)y_F - \left(p_H - t_H\right)y_H\theta$$

There are two possible cases: Case (a): $y_H \le y_F$ which leads to the case when $d_H + y_H \le d_H + y_F < d_F$:
$E\left[\Pi\left(p_H, X = d_H + y_F\right)\right] - E\left[\Pi\left(p_H, X = d_H + y_F\right)\right] = \left(y_F - y_H\right)\left[\left(p_H - t_H\right)\left(1+\theta\right) - c\right] \ge 0$ because $(y_H - y_F) > 0$ and $[(p_H - t_H)(1 + \theta) - c] > 0$ from Proposition 1 from the optimality condition of PHN policy.

Case (b): $y_F < y_H$ which leads to the case when $d_H + y_F < d_H + y_H < d_F$:
$E\left[\Pi\left(p_H, X = d_H + y_H\right)\right] - E\left[\Pi\left(p_H, X = d_H + y_F\right)\right] = \left(y_F - y_H\right)\left[p_H - c - t_H\right] > 0$ because $(y_H - y_F) < 0$ and $[p - c - t_H] < 0$ under PHN. Combining the two cases, we have $X^{PHN} = d_H + \max\{y_H, y_F\}$. Following a similar approach, it can be shown that $X^{PHN} = d_F + \max\{y_H, y_F\}$ when $d_F < d_H$. Thus, we can conclude that the optimal manufacturing quantity for the PHN policy under the minimum allocation requirement is $X^{PHN} = \min\{d_H, d_F\} + \max\{y_H, y_F\}$.❑

Proof of Proposition 8. Part (a): We complete the proof by comparing the expected profit under three cases: (1) manufacturing only in home country (i.e., $X_H = d_m$, $X_F = 0$), (2) manufacturing only in foreign country (i.e., $X_H = 0$, $X_F = d_m$), (3) manufacturing in both countries (i.e., $X_H = d_m - \delta$ $X_F = \delta > 0$). We show that the expected profit from manufacturing in both countries is dominated by manufacturing solely in one country (foreign country) for any given price. Because $p_H - t_H = p_F - t_F$, we simplify the notations for in-country revenue (i.e., price minus transportation cost) to $p - t$. We assume that international transshipment cost $t + \Delta$ is greater than the local transportation cost t with $\Delta > 0$. The following are the expected profit for the aforementioned three scenarios under the PHN policy:

$$E\left[\Pi\left(X_H = d_m, X_F = 0\right)\right] = \\ -cd_m + \int_0^{\frac{p-t-\Delta}{p-t}} \left(p-t\right) d_m f(e) de + \int_{\frac{p-t-\Delta}{p-t}}^{\frac{p-t}{p-t-\Delta}} \left(p-t\right) d_m f(e) de + \int_{\frac{p-t}{p-t-\Delta}}^{\infty} \left(p-t-\Delta\right) e d_m f(e) de \tag{28}$$

$$E\left[\Pi\left(X_H = 0, X_F = d_m\right)\right] = \\ -cd_m + \int_0^{\frac{p-t-\Delta}{p-t}} \left(p-t-\Delta\right) d_m f(e) de + \int_{\frac{p-t-\Delta}{p-t}}^{\frac{p-t}{p-t-\Delta}} \left(p-t\right) e d_m f(e) de + \int_{\frac{p-t}{p-t-\Delta}}^{\infty} \left(p-t\right) e d_m f(e) de \tag{29}$$

$$E\left[\Pi\left(X_H = d_m - \delta, X_F = \delta\right)\right] = -cd_m + \int_0^{\frac{p-t-\Delta}{p-t}} \left[\left(p-t\right)\left(d_m - \delta\right) + \left(p-t-\Delta\right)\delta\right] f(e) de \\ + \int_{\frac{p-t-\Delta}{p-t}}^{\frac{p-t}{p-t-\Delta}} \left[\left(p-t-\Delta\right)\left(d_m - \delta\right) + \left(p-t\right)\delta e\right] f(e) de + \int_{\frac{p-t}{p-t-\Delta}}^{\infty} \left[\left(p-t-\Delta\right)\left(d_m - \delta\right) e + \left(p-t\right)\delta e\right] f(e) de \tag{30}$$

We first compare (28) and (29) to determine the dominance between the two cases – manufacturing only in home country and manufacturing only in foreign country:

$$E\left[\Pi\left(X_H = 0, X_F = d_m\right)\right] - E\left[\Pi\left(X_H = d_m, X_F = 0\right)\right] \\ = \left(p-t\right) d_m \left\{ \Delta \left[\int_{\frac{p-t}{p-t-\Delta}}^{\infty} e f(e) de - \int_0^{\frac{p-t-\Delta}{p-t}} f(e) de \right] + \int_{\frac{p-t-\Delta}{p-t}}^{\frac{p-t}{p-t-\Delta}} \left(e-1\right) f(e) de \right\} \tag{31}$$

To determine the sign of (31), we examine each term in the curly bracket. In order to show that the first term is positive, we replace $\int_{\frac{p-t}{p-t-\Delta}}^{\infty} e f(e) de$ with a smaller value $\int_{\frac{p-t}{p-t-\Delta}}^{\infty} f(e) de$. Because $\int_{\frac{p-t}{p-t-\Delta}}^{\infty} f(e) de - \int_0^{\frac{p-t-\Delta}{p-t}} f(e) de = 1 - F\left(\frac{p-t}{p-t-\Delta}\right) - F\left(\frac{p-t-\Delta}{p-t}\right) > 0$, the first term is positive. The second term in the curly bracket is positive given that the pdf of the stochastic exchange rate is symmetric. To prove this, we measure the absolute difference between the two boundary points from the mean value and show that the absolute difference is greater for the upper bound compared to the lower bound. The difference between the upper bound and the mean equals to $\frac{p-t}{p-t-\Delta} - 1 = \frac{\Delta}{p-t-\Delta}$. The difference between the upper bound and mean equals to $1 - \frac{p-t-\Delta}{p-t} = \frac{\Delta}{p-t}$. Because $\frac{\Delta}{p-t-\Delta} - \frac{\Delta}{p-t} > 0$, the second term is positive. Thus, both terms in the curly bracket of (31) are positive, and we conclude that the expected profit for manufacturing only in foreign country dominates the expected profit manufacturing only in home country.

Next, we compare (29) and (30) in order to determine the dominance between the two cases – manufacturing only in foreign country and manufacturing in both countries:

$$E\left[\Pi\left(X_H = d_m, X_F = 0\right)\right] - E\left[\Pi\left(X_H = d_m - \delta, X_F = \delta\right)\right]$$
$$= \left(d_m - \delta\right)\left\{\Delta\left[\int_{\frac{p-t}{p-t-\Delta}}^{\infty} ef(e)de - \int_{0}^{\frac{p-t-\Delta}{p-t}} f(e)de\right] + \int_{\frac{p-t-\Delta}{p-t}}^{\frac{p-t}{p-t-\Delta}} \left[\left(p-t\right)e - \left(p-t-\Delta\right)\right] f(e)de\right\} \tag{32}$$

We examine each term in the curly bracket in order to determine the sign of (32). We know that the first term in the curly bracket is positive because it is identical with that of (31). It is clear that the second term in the curly bracket is positive. Because both terms in the curly bracket of (32) are positive, we conclude that the expected profit for manufacturing only in foreign country dominates that of manufacturing in both countries. Thus, under the PHN policy, manufacturing in both countries cannot be optimal, and is dominated by manufacturing solely in the foreign country.

Part (b): Part (a) of the proof identified that the optimal manufacturing takes place in a single country under the PHN policy when $\Delta > 0$. The proof for the optimal price to be below the total landed cost follows from Proposition 2 which has shown the result under a single manufacturing facility. ❑

Demand Uncertainty

The two market demand random error terms are expressed as $\tilde{z}_i$ with realizations z_i, and they are independently distributed with pdf $g_i(z_i)$ and cdf $G_i(z_i)$ for $i = H, F$. We begin our discussion by examining each market independently. If the firm charges a price higher than cost, according to PSNP, it would manufacture $X_H^* = G_H^{-1}((p_H - c - t_H)/p_H)$ for the home market and $X_F^* = G_F^{-1}((p_F - c - t_F)/p_F)$ for the foreign market. When the price is below cost, the optimal quantities correspond to the lower support of random demand values, i.e., $X_H^* = G_H^{-1}(0)$ and $X_F^* = G_F^{-1}(0)$.

The firm satisfies the VaR constraint that incorporates the probabilities from the three random variables $\left(\tilde{e}, \tilde{z}_H, \tilde{z}_F\right)$:

$$P_{\left(\tilde{e}, \tilde{z}_H, \tilde{z}_F\right)}\left[-cX + \pi^*\left(p_H, X, \tilde{e}\right) < -\beta\right] = \alpha\,. \tag{33}$$

Under demand uncertainty, the second-stage allocation problem can be expressed as follows:

$$\max_{\substack{(x_H, x_F)\geq 0 \\ x_H + x_F \leq X}} \pi\left(x_H, x_F \mid p_H, X, e\right) = \left[\begin{array}{l} \left(p_H - t_H\right)\int_0^{\infty} \min\left\{d_H\left(p_H, z_H\right), x_H\right\} g_H\left(z_H\right) dz_H \\ +\left(p_H - t_H\right)e\int_0^{\infty} \min\left\{d_F\left(p_H - t_H + t_F, z_F\right), x_F\right\} g_F\left(z_F\right) dz_F \end{array}\right] \tag{34}$$

Because the first-order derivatives with respect to allocation decisions are non-negative, i.e.,

$$\frac{\partial\pi\left(x_H, x_F \mid p_H, X, e\right)}{\partial x_H} = \left(p_H - t_H\right)\left[1 - G_H\left(x_H\right)\right] \geq 0, \text{ and } \frac{\partial\pi\left(x_H, x_F \mid p_H, X, e\right)}{\partial x_F} = \left(p_H - t_H\right)e\left[1 - G_F\left(x_F\right)\right] \geq$$

0, the value of both allocation decision variables would increase up to the point that $x_H + x_F = X$ as long as $X \leq G_H^{-1}(1) + G_F^{-1}(1)$. A Lagrangian approach can be employed in order to determine the optimal solution. We divide the demand probability space into five regions:

$P(\text{A}) = P[0 < X < \min\{d_H(p_H, z_H), d_F(p_H - t_H + t_F, z_F)\}]$,

$P(\text{B1}) = P[d_F(p_H - t_H + t_F, z_F) < X < d_H(p_H, z_H)]$,

$P(\text{B2}) = P[d_H(p_H, z_H) < X < d_F(p_H - t_H + t_F, z_F)]$,

$P(\text{C}) = P[\max\{ d_H(p_H, z_H), d_F(p_H - t_H + t_F, z_F)\} < X < d_H(p_H, z_H) + d_F(p_H - t_H + t_F, z_F)]$, and

$P(\text{D}) = P[X > d_H(p_H, z_H) + d_F(p_H - t_H + t_F, z_F)]$.

Let λ be the Lagrangian multiplier for the constraint $x_H + x_F \leq X$, and $E[\lambda_j]$ for j = A, B1, B2, C, D represent the expectation of the Lagrangian multiplier in the specified probability regions. $E[\lambda_j]$ is continuous and increasing in j = A, B1, B2, C, and is equal to zero in j = D. Thus, the optimal solution satisfies

$$E[\lambda(X)] = E[\lambda_{\text{A}}]\, P(\text{A}) + E[\lambda_{\text{B1}}]\, P(\text{B1}) + E[\lambda_{\text{B2}}]\, P(\text{B2}) + E[\lambda_{\text{C}}]\, P(\text{C}) = c. \tag{35}$$

The optimal allocation decisions can be classified in three regions. With smaller manufacturing quantities and low realizations of exchange rates, the allocation is made only to the home market, satisfying $G_H(x_H) = (p_H - t_H - \lambda)/(p_H - t_H)$ or $\lambda = (p_H - t_H)(1 - G_H(x_H)) \equiv \lambda(x_H)$. With smaller manufacturing quantities but higher realizations of exchange rates, the allocation is made only to the foreign market, satisfying $G_F(x_F) = ((p_H - t_H)e - \lambda)/((p_H - t_H)e)$ or $\lambda = (p_H - t_H)e(1 - G_F(x_F)) \equiv \lambda(x_F)$. In the rest of the scenarios, allocation occurs to both markets satisfying $G_H^{-1}((p_H - t_H - \lambda)/(p_H - t_H)) + G_F^{-1}(((p_H - t_H)e - \lambda)/((p_H - t_H)e)) = X$. Figure A1 shows how the optimal second-stage decisions vary according to the realized value of the exchange rate e for a given pair of (p_H, X) for independent uniform demand distributions at each market. In Figure A1, the curve L1 is obtained by solving $1 - G_H(X) = e(1 - G_F(0)) = e$, curve L2 is obtained by solving $1 - G_H(0) = 1 = e(1 - G_F(X))$ at each exchange-rate realization e.

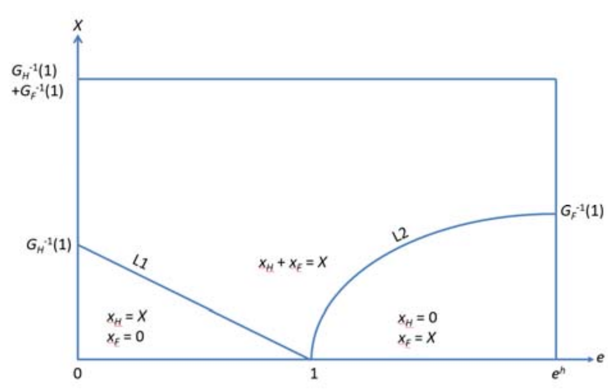

Figure A1. Optimal second-stage decisions for a given p_H and X and uniformly distributed random demand.

In PSNP, additive demand random error term is known to reduce the optimal selling price relative to deterministic demand (Petruzzi and Dada 1999). This same effect can be seen in our model with exchange-rate uncertainty. Compared to the case of deterministic demand, profit decreases because of increased uncertainty due to stochastic demand, and the optimal price is lower (given an additive demand random error term). A reduction in the manufacturing quantity is also caused by an increased level of risk stemming from greater quantity-demand mismatches. Thus, the firm acts even more conservatively and reduces its production amount below what it used to manufacture under deterministic demand.

In sum, while demand uncertainty in isolation does not lead to pricing below cost, the behavior can be observed in the presence of combined exchange-rate and demand uncertainty. Moreover, when demand uncertainty is introduced in the form of an additive error term, it would lead to a further reduction in price, making the pricing below cost result even more pronounced under stochastic demand.