Wine Analytics: Fine Wine Pricing and Selection under Weather and Market Uncertainty

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We examine a risk-averse distributor's decision in selecting between bottled wine and wine futures under weather and market uncertainty. At the beginning of every summer, a fine wine distributor has to choose between purchasing bottled wine made from the harvest collected two years ago and wine futures of wine still aging in the barrel from the harvest of the previous year. At the end of the summer, after realizing weather and market fluctuations, the distributor can adjust her allocation by trading futures and bottles.

The paper makes three contributions. First, we develop an analytical model in order to determine the optimal selection of bottled wine and wine futures under weather and market uncertainty. Our model is built on an empirical foundation in which the functional forms describing the evolution of futures and bottle prices are derived from comprehensive data associated with the most influential Bordeaux winemakers. Second, we develop structural properties of optimal decisions. We show that a wine distributor should always invest in wine futures because it increases the expected profit in spite of being a riskier asset than bottled wine. We characterize the influence of variation in various uncertainties in the problem. Third, our study empirically demonstrates the financial benefits from using our model for a large distributor. The hypothetical average profit improvement in our numerical analysis is significant, exceeding 21%, and its value becomes higher under risk aversion. The analysis is beneficial for fine wine distributors as it provides insights into how to improve their selection in order to make financially healthier allocations.

Keywords: *wine futures, pricing, weather uncertainty, market uncertainty, risk aversion*

1. Introduction

This paper examines a wine distributor's annual decision regarding the selection of bottled wine and wine futures under weather and market uncertainty. At the end of each summer, a winemaker harvests grapes, crushes them in order to produce wine. A fine wine goes through a long aging process ranging between 18 to 24 months. The wine can be sold in advance in the form of wine futures, often referred to as "en primeur" due to the popular futures campaign for Bordeaux wines. Wine futures begin to trade before the first summer following the harvest (approximately eight months after harvest). The wine gets bottled in the second summer and is sold for retail and distribution; those who purchased this wine in the form of futures also receive their wine shipment.

To understand the difference between bottled wine and wine futures, let us consider the 2013 vintage of a fine wine as an example provided in Figure 1. The 2013 vintage of this wine is made from the grapes harvested in September 2013; its futures are sold in May 2014, and the wine is bottled and sold in May 2015. Similarly, the 2014 vintage is produced from the grapes harvested in September 2014, and its futures come out in May 2015. As a result, the distributor has two products in May 2015 from the same fine wine producer: (1) The 2013 vintage in the form of bottled wine, and (2) the 2014 vintage in the form of wine futures (a contract to take the possession of the 2014 vintage wine in May 2016). Thus, in May

2015, a fine wine distributor has to select the amounts of bottled wine from the 2013 vintage and wine futures of the 2014 vintage. A distributor's business involves buying the wine from the winemaker and immediately pushing it downstream to the wholesalers and retail stores. Thus, its profits are based on quick movement of wine, rather than opportunistic sale based on wine prices. Our paper assists wine distributors by developing an analytical model to determine the allocation decisions between bottled wine and wine futures under weather and market uncertainty. The model relies on an empirical foundation that describes the price evolution of futures and bottles. The empirical analysis provides the justification for the functional forms describing the impact of weather and market conditions on prices.

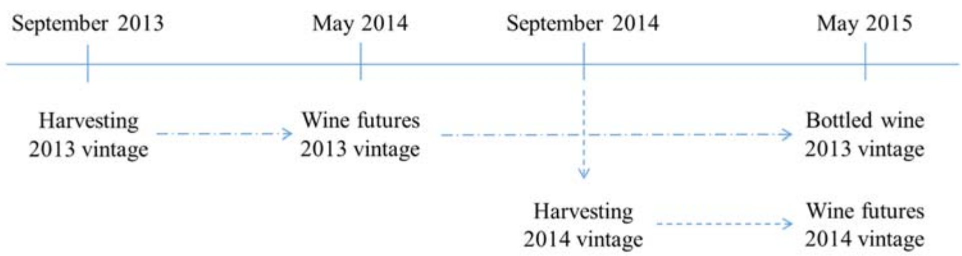

Figure 1. The timeline of futures and bottle trade in wine production.

Quality of a fine wine is greatly influenced by weather conditions during the grape growing season; often higher temperatures lead to better quality of grapes and wine. Due to differences in weather conditions from one year to the other, two consecutive vintages of the same wine may have very different quality, and hence, price. A striking example regarding the impact of weather on wine futures prices can be seen from the Bordeaux region where the summer of 2005 was very hot and dry, resulting in one of the finest vintages in recent years. Prior to the growing season in 2005, the wine futures for the 2004 vintage of Troplong Mondot was released to the market at the price of \$62/bottle. The impact of superior weather in the summer of 2005 was so big that the wine futures price for the 2005 Troplong Mondot jumped to \$233/bottle, corresponding to a 276% increase when compared with the futures price of the previous vintage. This is an example of the improved weather conditions from 2004 to 2005, and its impact on wine futures prices. Moreover, the positive weather during the summer of 2005 negatively impacted the 2004 vintage wine, and caused the bottle price of the 2004 vintage to go down to \$54 per bottle, resulting in a 13% reduction from its futures price from the prior year. This is an example where the growing weather condition not only influences the wine futures price of its vintage but also the evolution of a futures price to the bottle price in the previous vintage.

In addition to weather fluctuations, changes in the market conditions also drive fine wine prices. All fine wine futures and bottles are traded in London International Vintner's Exchange (Liv-ex) with standardized contracts. We use Liv-ex 100 index, composed of 100 most sought-after wines, in order to describe the fine wine market conditions. This index is declared as the "fine wine industry's leading benchmark" by Reuters. When Liv-ex 100 index decreased by 17.17% in 2008 (in comparison to 2007), the top Bordeaux winemakers priced their 2008 vintage wines 16.66% less than their 2007 vintage wines on average despite the highly similar weather conditions between the two growing seasons. Our analysis combines the impact of weather and market fluctuations in explaining the price evolution of wine futures and bottled wine. These price evolution functions are utilized in developing an analytical model to help the distributor's selection between wine futures and bottled wine.

Wine distribution is an important business around the world. In the US alone, the wine industry generates $37.6 billion each year with a projected 8.2% growth in the upcoming years. Under the presence of drastic changes in vintage prices depending on weather and market conditions, a wine distributor is often puzzled with whether to invest in wine futures of the previous year's vintage or buy recently bottled wine from two vintages ago. While wine futures exhibit a greater uncertainty as future weather conditions can negatively influence the bottle price as in the example of the 2004 Troplong Mondot, it also allows the distributor to lock up limited supply at lower prices. Moreover, futures can be easily traded in Liv-ex, the exchange platform for fine wine without having to make physical shipments and comply with legal restrictions. Thus, wine futures are highly liquid in comparison to bottled wine. Purchasing bottles can be perceived as a safer bet upfront as the bottle prices are revealed. However, market conditions continue to influence these prices. The distributor can observe the summer weather conditions getting comparative indications as to how the futures price is going to evolve to the bottle price. Moreover, the distributor can later change its allocation through buying additional or selling existing futures with limited ability to move its bottled wine inventory.

When should a wine distributor engage in futures? Our work finds motivation from conversations with the executives at the largest wine distributor in the US and in the world that does not invest in wine futures due to the lack of knowledge about futures prices and their evolution to bottle prices. Earlier research (Ashenfelter et al. 1995 and Ashenfelter 2008) has shown that mature Bordeaux wine prices can be predicted accurately using growing season weather conditions, but these studies conclude that young wine prices (i.e., futures prices and prices for the recently released bottled wines) cannot be predicted using weather conditions. Our empirical analysis provides an explanation for the impact of weather and market changes in young wine prices. It serves as a foundation for our analytical model, and enables us to estimate the distributor's economic benefit from investing in a combination of wine futures and bottled wine (when compared with a distributor that invests only in bottled wine).

Wine futures are often perceived to be a riskier alternative than bottled wine. Our empirical analysis confirms this perception as it shows that wine futures prices are influenced by both weather and market fluctuations, whereas bottled wine prices are influenced only by the changes in market conditions. Thus, a distributor would not be encouraged to make investments in futures. Rather, the distributor would spend its money in physical bottles where the price is already evolved and has smaller uncertainty. Indeed, this has been the practice at some of the distributors as they invest solely in bottled wine, bypassing the futures alternative. Our analytical model shows, however, that a distributor should always make some investment in futures. This finding is confirmed through a numerical analysis using comprehensive data.

The remainder of the paper is organized as follows. Section 2 reviews the relevant literature from economics, operations and supply chain management, and demonstrates how our work differs from earlier publications. Section 3 develops an analytical model to help a distributor determine the allocation decisions between wine futures and bottled wine. Section 4 presents the economic benefit from our proposed model using comprehensive data from the most influential Bordeaux winemakers. Section 5 presents our conclusions and managerial insights. All proofs and derivations, and the details of our data collection are presented in the appendix.

2. Literature Review

The economics literature has shown significant interest in understanding, explaining, and predicting wine prices. Ashenfelter et al. (1995) and Ashenfelter (2008) are the two seminal papers showing that mature Bordeaux wine prices can be predicted using weather and age with accuracy, however, they both conclude that their models fail to explain young wine prices. For a wine distributor, however, most trade takes place when the wine is young, and therefore, it is important to understand the evolution of young wine prices. Our work examines how young wine prices are impacted by the fluctuations in weather and market conditions. While we complete this analysis in order to build an analytical model that determines the optimal selection of wine futures and bottled wine, our empirical findings complement earlier publications by providing an explanation for the evolution of young wine prices.

Jones and Storchmann (2001), Lecocq and Visser (2006), Ali and Nauges (2007), Ali et al. (2008), and Ashenfelter and Jones (2013) also address the price prediction of Bordeaux wines based on weather conditions and/or tasting scores. Byron and Ashenfelter (1995) and Wood and Anderson (2006) extend this stream to Australian wines while Haeger and Storchmann (2006) and Ashenfelter and Storchmann (2010) examine American wines and German wines, respectively. However, none of these papers focus on young wine pricing nor have a selection analysis that can benefit distributors.

Noparumpa et al. (2015) investigate the impact of tasting scores on young wine prices, and then provide a model for winemakers to determine the optimal amount of wine to be sold in the form of futures and the optimal amount that should be sold after the wine is bottled. Their work concludes that wine futures help a winemaker collect her revenues in advance while passing the risk of having a poor quality vintage to the distributor. They estimate that selling wine in advance in the form of futures increases

Bordeaux winemakers' profits by 10% on average. If winemakers are the clear winners of futures trade, then one asks what is in it for the wine distributors. Our paper sheds light on this question by providing an analytical model which incorporates the advantages (i.e., being easily tradable through the Liv-ex platform) and the disadvantages (i.e., bearing a greater price uncertainty) of wine futures for distributors. We utilize weather and market fluctuations instead of tasting scores (correlated with weather) to explain futures prices; this leads to considerably higher explaining power with greater adjusted R^2 values in a larger sample featuring the leading Bordeaux winemakers. Moreover, our explanation of the evolution of a futures price into bottle price is a unique aspect of our study.

Wine futures is a form of advance selling and purchasing, and recent publications advocate the use of advance selling in various settings. Xie and Shugan (2001) exemplify the benefits in electronic tickets and online platforms. Cho and Tang (2013) examine the influence of supply and demand uncertainty, and Tang and Lim (2013) investigate the influence of speculators in advance selling. Boyacı and Özer (2010) demonstrate the advantages of advance selling in capacity planning. Our work departs from these studies in three features: (1) The wine distributor has to choose between advance purchase of an upcoming product in replacement of the present product; (2) as the price evolves through revelations of uncertainty, the distributor has the ability to adjust its selection between the two product offerings; (3) the sources of uncertainty in our problem are weather and market fluctuations differentiating our problem setting.

Wine futures depart from the commodity futures described in Fama and French (1987) and Geman (2005). In commodity markets (e.g., corn, soybean, cocoa), a settlement in a futures contract means that the agricultural product delivered to the buyer can be produced by any farmer. In fine wine, however, if a buyer is asking for a bottle of 2008 Lafite Rothschild, the seller cannot substitute it with a bottle of 2007 Lafite Rothschild, or a bottle of 2008 Troplong Mondot. Thus, fine wine cannot be substituted across producers or vintages, and therefore, is not a commodity. Moreover, in traditional commodities, futures contracts and spot purchases occur simultaneously for the commodity product. However, spot purchases of bottled fine wine do not begin until the completion of the futures trade of the same wine.

Fine wines are also treated as a long-term investment. Storchmann (2012) provides a comprehensive review about wine economics, and covers the use of wine as an investment option. Dimson et al. (2014) find that young Bordeaux wines yield greater returns than the mature ones. This finding further amplifies the importance of explaining the evolution of young wine prices. Jaeger (1981), Burton and Jacobsen (2001), and Masset and Weisskopf (2010) also examine the return on wines as a long-term investment. Jaeger (1981), Burton and Jacobsen (2001), and Dimson et al. (2014) conclude that wines can yield greater returns than treasury bills, but less than equities. Masset and Weisskopf (2010), on the other hand, demonstrate that fine wines can outperform equities during a financial crisis when financial assets are highly correlated. While these studies consider wine as a long-term investment, our paper focuses on the benefits as a short-term investment from a distributor's perspective who buys the recently released young wines from winemakers and sells to the wholesalers and retailers shortly after.

Supply uncertainty is another related stream as quality and price may vary dramatically across different vintages of the same wine depending on weather and market conditions. Yano and Lee (1995) provide a comprehensive review of the literature that focuses on supply uncertainty as a consequence of yield fluctuations. Jones et al. (2001) examine the impact of yield uncertainty in the corn seed industry for a firm that utilizes farmland in two opposing hemispheres, and develop a two-stage production scheme to better match supply and demand. Kazaz (2004) introduces the impact of yield fluctuations into what he defines as the yield-dependent cost and price structures in the olive oil industry. Kazaz and Webster (2011) add a price-setting capability, and show how yield fluctuations influence a firm's pricing decisions. Their study also demonstrates the benefits of using fruit futures (if existed) in mitigating supply uncertainty. Boyabatli et al. (2011) and Boyabatli (2015) examine the purchasing contracts for fixed-proportion technology products in the presence of random spot prices. Kazaz and Webster (2015) develop optimal price and quantity decisions under supply and demand uncertainty and under risk aversion. Tomlin and Wang (2008) develop price and quantity decisions in a co-production setting that results from random yield in the split of two distinct products. Li and Huh (2011) also develop price and quantity decisions for multiple products using a multinomial logit model. Departing from these papers, we define supply uncertainty in the form of variation in quality due to growing season weather; hence, wine futures have a quality-dependent price structure. Moreover, the secondary (emergency) investment option utilized in some of these papers becomes available in the second stage whereas, in our model, both wine futures and bottled wines are simultaneously available at the beginning.

Weather and market realizations provide signals to the wine industry, and the impact of similar signals, in particular for estimating demand, is examined widely in the operations management literature. Gümüş (2014), for example, investigates the impact of forecast as a signal for demand. Our work departs from this body of literature as we study signals that influence the evolution of price over time.

3. The Model and its Analysis

This section develops and analyzes a model that helps the wine distributor determine the investment allocation between wine futures and bottled wine. The prices of wines futures and bottled wine are influenced by the randomness in weather and market conditions after these decisions take place. In this model, the functional forms describing the evolution of futures and bottle prices rely on an empirical foundation.

In each May, a risk-averse wine distributor has to select between wine futures (of new vintage) and bottled wine (of previous vintage) of a winemaker. Specifically, in May of calendar year t, the distributor has to determine the amount of money to be invested in wine futures from vintage $t-1$ and bottled wine from vintage $t-2$.

3.1. Empirical Foundation for the Model

In this section, we present an empirical analysis that serves as a foundation for our mathematical model that will be presented in Section 3.2. The results of our empirical analysis determines the functional forms describing the price evolution of young wines as functions of weather and market random variables. The functional forms that emerge from the empirical analysis are used in the analytical model.

We begin our discussion with realized values of futures and bottled wine prices. In May of calendar year t, futures for vintage $t-1$ are released at the (realized) price of $rf_1^{j,t-1}$ for winemaker j. We express the futures price of the same vintage for winemaker j in September of calendar year t as $rf_2^{j,t-1}$, and in May of calendar year $t+1$ as $rf_3^{j,t-1}$. In May of calendar year t, bottled wine of winemaker j from vintage $t-2$ is also released, and we express this (realized) bottle price as $rb_1^{j,t-2}$. We denote the bottle price of vintage $t-2$ from winemaker j in September of calendar year t with $rb_2^{j,t-2}$, and in May of calendar year $t+1$ with $rb_3^{j,t-2}$. Figure 2 illustrates the evolution of realized futures and bottle prices over time.

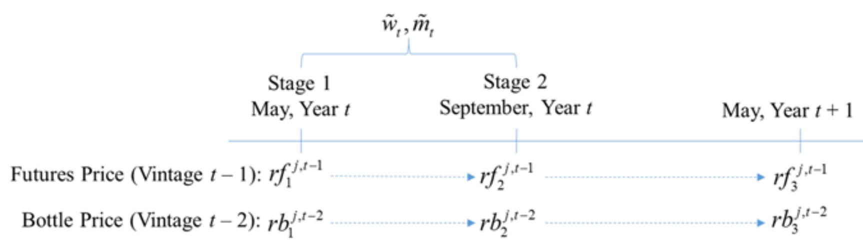

Figure 2. The evolution of futures and bottled wine prices under weather and market uncertainty.

After the wine distributor makes investments in futures of vintage $t-1$ and bottled wine of vintage $t-2$ from winemaker j in May of calendar year t, a new summer weather information becomes available in calendar year t. This new summer weather information, which is fully observed by September of calendar year t, provides a relative comparison to the wines that are from vintages $t-1$ and $t-2$. For the case of wine futures of vintage $t-1$, the new weather information from May–September period of year t compared to the growing season of grapes (i.e., May–September period of year $t-1$) can play a role. Thus both $rf_2^{j,t-1}$ and $rf_3^{j,t-1}$ can be influenced by the new weather information. For the case of bottled wine of vintage $t-2$, the new weather information from May–September period of year t compared to the growing season of grapes (i.e., May–September period of year $t-2$) can also influence the values of $rb_2^{j,t-2}$ and $rb_3^{j,t-2}$. Similarly, market conditions change from May to September of year t. As a consequence, the weather and market information observed at the end of summer in calendar year t can have an impact of the values of $rf_2^{j,t-1}$, $rf_3^{j,t-1}$, $rb_2^{j,t-2}$, and $rb_3^{j,t-2}$.

We next examine the impact of weather and market fluctuations on the evolution of wine futures and bottled wine prices. Let us denote weather fluctuations with random variable $\tilde{w}_t$ and its realization with w_t, and we denote market fluctuations with random variable $\tilde{m}_t$ and its realization with m_t. The results provide justification for the functional forms of futures and bottled wine prices in our analytical model as functions of w_t and m_t. The appendix provides a detailed explanation of the data used in the empirical analysis. The empirical analyses utilize standardized prices of wine futures and bottled wine as in Noparumpa et al. (2015).

3.1.1. Models 1A and 1B: Futures Price Evolution

We express the standardized futures price of vintage $t-1$ from winemaker j in stage $i = \{1, 2, 3\}$ as $sf_i^{j,t\text{-}1} = (rf_i^{j,t-1} - \mu_{f_i^j})/\sigma_{f_i^j}$ where $\mu_{f_i^j}$ and $\sigma_{f_i^j}$ represent the mean and the standard deviation of the futures price.

For the futures of vintage $t-1$, we denote the average temperature difference between the new growing season (of calendar year t) and the wine's own growing season by w_t. A positive (negative) w_t implies that the new growing season is relatively warmer (colder) than the growing season of the futures. Our choice of an absolute weather change measure (as opposed to percentage change) is consistent with Ashenfelter (2008) who uses an absolute measure of weather in his analysis. Unlike temperature, which conforms to a range that is relatively universal over each season, market indices may grow and shrink significantly over time, and thus percentage change is a more meaningful indicator than absolute change. We denote the percentage change in Liv-ex 100 index over the new growing season (of calendar year t) by m_t. A positive (negative) m_t implies that the market conditions are improved (worsened) over the new growing season.

We develop the following linear regression models designated as Model 1A and Model 1B, respectively, where $t = \{2008, 2009, 2010, 2011, 2012\}$ and $j = \{1, 2, \ldots, 44\}$:

$$(sf_2^{j,t\text{-}1} - sf_1^{j,t\text{-}1}) = \gamma_0 + \gamma_1 w_t + \gamma_2 m_t + \varepsilon_{j,t} \quad (1)$$

$$(sf_3^{j,t\text{-}1} - sf_2^{j,t\text{-}1}) = \eta_0 + \eta_1 w_t + \eta_2 m_t + \varepsilon_{j,t}. \quad (2)$$

Table 1 provides the regression analysis of the impact of new summer weather and market information on the price evolution of futures with $sf_2^{j,t\text{-}1}$ (in Model 1A) and $sf_3^{j,t\text{-}1}$ (in Model 1B).

The analysis in Table 1 provides four results. First, better weather of the upcoming vintage (i.e., higher value of w_t) has a negative impact on the evolution of futures price from $sf_1^{j,t\text{-}1}$ to $sf_2^{j,t\text{-}1}$. This weather effect is statistically significant at 1% level. This can be easily understood as the upcoming vintage had better weather conditions than the vintage of futures, and therefore, the price of wine futures would decrease. Moreover, better weather of the upcoming vintage (i.e., higher value of w_t) has a continued negative impact (statistically significant at 1%) on the evolution of futures price from $sf_2^{j,t\text{-}1}$ to $sf_3^{j,t\text{-}1}$. This implies that the new weather information is not completely priced in the futures as of

September of calendar year t. A similar observation is made in Ashenfelter (2008). Second, the negative coefficient representing the impact of weather in the evolution of futures price from $sf_2^{j,t-1}$ to $sf_3^{j,t-1}$ is greater in absolute value than that of $sf_1^{j,t-1}$ to $sf_2^{j,t-1}$. Third, improving market conditions during the summer of calendar year t (with a higher value of m_t) has a positive impact on the evolution of futures price both from $sf_1^{j,t-1}$ to $sf_2^{j,t-1}$ and from $sf_2^{j,t-1}$ to $sf_3^{j,t-1}$. This market effect is statistically significant at 1% level. Fourth, the positive coefficient representing the impact of market conditions in the evolution of futures price from $sf_2^{j,t-1}$ to $sf_3^{j,t-1}$ is greater than that of $sf_1^{j,t-1}$ to $sf_2^{j,t-1}$.

	Model 1A: $sf_2^{j,t-1} - sf_1^{j,t-1}$		**Model 1B:** $sf_3^{j,t-1} - sf_2^{j,t-1}$	
Parameter	Coefficient	*t-stat*	Coefficient	*t-stat*
Intercept	0.0296	*2.85****	0.0788	*4.45****
w_t	-0.0501	*-4.58****	-0.1281	*-6.88****
m_t	0.0079	*5.47****	0.0223	*9.01****
Adjusted R^2	0.19		0.37	
Observations	220		220	

Table 1. Linear regression results demonstrating the impact of weather and market conditions on the evolution of futures prices. *** denotes statistical significance at 1%.

3.1.2. Models 2A and 2B: Bottle Price Evolution

We express the standardized bottle price of vintage $t-2$ from winemaker j in stage $i = \{1, 2, 3\}$ as $sb_i^{j,t-2} = (rb_i^{j,t-2} - \mu_{b_i^j})/\sigma_{b_i^j}$ where $\mu_{b_i^j}$ and $\sigma_{b_i^j}$ represent the mean and the standard deviation of the bottle price.

For the bottles of vintage $t-2$, we denote the average temperature difference between the new growing season (of calendar year t) and the wine's own growing season by w_t. A positive (negative) w_t implies that the new growing season is relatively warmer (colder) than the growing season of the bottles.

We denote the percentage change in Liv-ex 100 index over the new growing season (of calendar year t) by m_t. A positive (negative) m_t implies that the market conditions are improved (worsened) over the new growing season.

We develop the following linear regression models designated as Model 2A and Model 2B, respectively, where $t = \{2008, 2009, 2010, 2011, 2012\}$ and $j = \{1, 2, \ldots, 44\}$:

$$(sb_2^{j,t-2} - sb_1^{j,t-2}) = \theta_0 + \theta_1 w_t + \theta_2 m_t + \varepsilon_{j,t} \quad (3)$$

$$(sb_3^{j,t-2} - sb_2^{j,t-2}) = \lambda_0 + \lambda_1 w_t + \lambda_2 m_t + \varepsilon_{j,t}. \quad (4)$$

Table 2 provides the regression analysis of the impact of new summer weather and market information on the evolution of bottle prices described as $sb_2^{j,t-2}$ (in Model 2A) and $sb_3^{j,t-2}$ (in Model 2B).

	Model 2A: $sb_2^{j,t-2} - sb_1^{j,t-2}$		**Model 2B:** $sb_3^{j,t-2} - sb_2^{j,t-2}$	
Parameter	Coefficient	*t-stat*	Coefficient	*t-stat*
Intercept	0.0248	*1.52*	0.0187	*0.53*
w_t	-0.0082	*-0.59*	0.0245	*0.82*
m_t	0.0059	*2.19***	0.0255	*4.43****
Adjusted R^2	0.01		0.12	
Observations	220		220	

Table 2. Linear regression results demonstrating the impact of weather and market conditions on the evolution of bottle prices. ** and *** denote statistical significance at 5% and 1%, respectively.

The analysis in Table 2 provides three results. First, weather conditions of the upcoming vintage (i.e., the value of w_t) does not have a statistically significant effect on the evolution of bottle prices, neither from $sb_1^{j,t-2}$ to $sb_2^{j,t-2}$, nor from $sb_2^{j,t-2}$ to $sb_3^{j,t-2}$. Second, improving market conditions during the summer of calendar year t (with a higher value of m_t) has a positive impact on the evolution of bottle prices both from $sb_1^{j,t-2}$ to $sb_2^{j,t-2}$ and from $sb_2^{j,t-2}$ to $sb_3^{j,t-2}$. This market effect is statistically significant at 5% level in Model 2A and 1% level in Model 2B. Third, the positive coefficient representing the impact of market conditions in the evolution of futures price from $sb_2^{j,t-2}$ to $sb_3^{j,t-2}$ is greater than that of $sb_1^{j,t-2}$ to $sb_2^{j,t-2}$.

3.1.3. Functional Forms for the Analytical Model

We next present the functional forms that emerge from the empirical analysis and that will be used to describe the price evolution of wine futures and bottled wine as functions of weather and market uncertainty. We drop the superscripts j, $t - 1$, and $t - 2$ from futures and bottled wine prices, and the subscript t from w and m for notational simplicity because the analytical model given in Section 3.2 examines the distributor's investment decision in futures of vintage $t - 1$ and bottles of vintage $t - 2$ of a single winemaker (i.e., an arbitrary j) in May of an arbitrary year t.

We begin with the functional forms representing the price evolution of wine futures. We denote the futures price in May of calendar year t as f_1. For a given (w, m), we define the functional form of the realized futures price in September of calendar year t with $f_2(w, m)$, and the (expected) futures price in May of calendar year $t + 1$ with $f_3(w, m)$.

We use the four empirical findings in Section 3.1.1 regarding the impact of weather and market fluctuations on futures prices in order to describe the functional forms. In the first empirical finding, the coefficients of weather random variable are negative, and therefore we define $\partial f_2(w, m)/\partial w < 0$ and $\partial f_3(w, m)/\partial w < 0$. In the second empirical finding, the negative coefficient representing the impact of weather in the evolution of futures price from $sf_2^{j,t-1}$ to $sf_3^{j,t-1}$ is greater in absolute value than that of $sf_1^{j,t-1}$ to $sf_2^{j,t-1}$. Therefore, we define the functional form of the futures price evolution as $\partial f_3(w, m)/\partial w < \partial f_2(w, m)/\partial w < 0$. In the third empirical finding, the coefficients of market random variable are positive, and therefore, we define $\partial f_2(w, m)/\partial m > 0$ and $\partial f_3(w, m)/\partial m > 0$. In the fourth empirical finding, the positive coefficient representing the impact of market conditions in the evolution of futures price from $sf_2^{j,t-1}$ to $sf_3^{j,t-1}$ is greater than that of $sf_1^{j,t-1}$ to $sf_2^{j,t-1}$. Therefore, we define the functional form of the futures price evolution as $\partial f_3(w, m)/\partial m > \partial f_2(w, m)/\partial m > 0$.

We next present the functional forms that describe the price evolution of bottled wine. We denote the bottle price in May of calendar year t as b_1. We use the three empirical findings in Section 3.1.2 regarding the impact of weather and market fluctuation on bottled wine prices in order to describe the functional forms. Because the first empirical finding indicates that weather is not statistically significant in the evolution of bottle price, the functional forms representing bottle prices do not feature w in their arguments. For a given (w, m), we define the functional form of the realized bottle price in September of calendar year t with $b_2(m)$, and the (expected) bottle price in May of calendar year $t + 1$ with $b_3(m)$. In the second empirical finding, the coefficients of market random variable are positive, and therefore, we define $\partial b_2(m)/\partial m > 0$ and $\partial b_3(m)/\partial m > 0$. In the third empirical finding, the positive coefficient representing the impact of market conditions in the evolution of futures price from $sb_2^{j,t-2}$ to $sb_3^{j,t-2}$ is greater than that of $sb_1^{j,t-2}$ to $sb_2^{j,t-2}$. Therefore, we define the functional form of the bottle price evolution as $\partial b_3(m)/\partial m > \partial b_2(m)/\partial m > 0$.

Our empirical analyses employ standardized prices. It is important to note that when the regression analyses presented in tables 1 and 2 are replicated using natural logarithm of prices, we obtain similar results. However, our results obtained with standardized prices might show some bias if we were to use a longer panel data. In order to prevent the potential bias, one can split the data into two subsets by the longitudinal dimension, then use the first set to compute the values of mean and standard deviation, and use the second set to replicate the analyses using the values of mean and standard deviation obtained from the first set (not practical in our setting due to limited time frame of data).

3.2. The Model

We formulate the distributor's problem using a two-stage stochastic program with recourse. In stage 1 (May of year t), the distributor determines the investment in futures of vintage $t - 1$ (denoted x_1) and bottles of vintage $t - 2$ (denoted y_1) of a single winemaker, respectively, with a limited budget (denoted B) and a value-at-risk (VaR) constraint. Distributors have a well-specified budget for each fine winemaker, and executives describe their risk tolerance in the form of a VaR constraint. Recall that f_1 and b_1 are the unit price of futures and bottles in stage 1. For notational simplicity in this section, we normalize $f_1 = b_1 = 1$ without loss of generality. At the end of stage 1 (September of year t), the distributor observes the realization (w, m) of weather and market random variables. We normalize the means to zero, i.e., $E[\tilde{w}] = E[\tilde{m}] = 0$. The probability density functions (pdf) of $\tilde{w}$ and $\tilde{m}$ are denoted $\phi_w(w)$ and $\phi_m(m)$ on respective support $[w_L, w_H]$ and $[m_L, m_H]$. We let $\Omega = [w_L, w_H] \times [m_L, m_H]$.

At the beginning of stage 2 (September of year t), the distributor determines the amount of futures to buy or sell (denoted x_2) at price $f_2(w, m)$, and the amount of bottles to purchase (denoted y_2) at price $b_2(m)$. The distributor can easily buy or sell futures by transferring the ownership rights through Liv-ex; the transaction does not require any physical flow of good and is not subject to any legal requirements. However, while the distributor can purchase bottles from the winemaker, the selling of bottles faces logistical and legal constraints. First, Bordeaux winemakers prefer shipping the bottled wine in the winter months to prevent any deterioration during transportation. Consequently, the bottles purchased in May of year t (stage 1) are not in distributor's possession as of September of year t (stage 2). Hence, she cannot sell those bottles immediately at the beginning of stage 2. Second, selling a bottle to a different owner has legal constraints in the US where the sale of the bottle from one distributor located in another state can be considered as illegal movement of spirits. The combination of these two facts restrict the distributor from selling the bottled wine in September of year t (stage 2); these bottles are directly sold to the customers of the distributor (wholesalers, liquor stores, and consumers) at the end of stage 2. However, the distributor can buy additional bottles from the winemaker using either the cash leftover from stage 1 or from the sale of futures.

At the end of stage 2, the distributor collects revenues from futures (that are bottled by then) and bottles. Futures and bottle prices at the end of stage 2 are also uncertain. The uncertainty in futures price between September of year t and May of year $t + 1$ is captured by random variable $\tilde{z}_f$. The realized futures price is $f_3(w, m) + z_f$. The uncertainty in bottle price between September of year t and May of year $t + 1$ is captured by random variable $\tilde{z}_b$. The realized bottle price is $b_3(m) + z_b$. We assume that $(\tilde{z}_f, \tilde{z}_b)$ is independent of $(\tilde{w}, \tilde{m})$, and have a mean of zero, i.e., $E[\tilde{z}_f] = E[\tilde{z}_b] = 0$. Thus, $E[f_3(w, m) + \tilde{z}_f] = f_3(w, m)$ and $E[b_3(m) + \tilde{z}_b] = b_3(m)$. By examining our price data, we observe that if futures (bottle) price moves in one direction when it evolves from f_1 to $f_2(w, m)$ (from b_1 to $b_2(m)$), then a wide majority of realized futures (bottle) prices at the end of stage 2 also move in the same direction when they evolve from $f_2(w, m)$ to $f_3(w, m) + z_f$ (from $b_2(m)$ to $b_3(m) + z_b$). We insert the following assumptions that comply with this observation:

$$\text{If } f_2(w, m) \Diamond f_1 \text{, then } E[f_3(w, m) + \tilde{z}_f] \Diamond f_2(w, m) \text{ for all } \Diamond \in \{>, =, <\} \text{ and for all } (w, m). \tag{5}$$

If $b_2(m) \Diamond b_1$, then $E[b_3(m) + \tilde{z}_b] \Diamond b_2(m)$ for all $\Diamond \in \{>, =, <\}$ and for all m. (6)

All price functions $f_2(w, m)$, $f_3(w, m)$, $b_2(m)$ and $b_3(m)$, are linear in their arguments, and are net of transaction, shipping, and other costs, i.e., the prices reflect the net revenues in these two stages. Thus, the realized profit at the end of stage 2 can be expressed as follows:

$$\Pi(x_1, y_1, w, m, x_2, y_2, z_f, z_b)$$
$$= -x_1 - y_1 - f_2(w, m)x_2 - b_2(m)y_2 + [f_3(w, m) + z_f](x_1 + x_2) + [b_3(m) + z_b](y_1 + y_2). \tag{7}$$

At the beginning of stage 2, the distributor selects x_2 and y_2 to maximize expected recourse profit subject to budget and VaR constraints given the initial investments in futures and bottles (x_1, y_1) and the realized values of weather and market random variables (w, m):

$$\max_{x_2, y_2} E\left[\Pi\left(x_1, y_1, w, m, x_2, y_2, \tilde{z}_f, \tilde{z}_b\right)\right] \tag{8}$$

subject to

$$f_2(w, m)x_2 + b_2(m)y_2 \le B - x_1 - y_1 \tag{9}$$

$$P\left[\Pi\left(x_1, y_1, w, m, x_2, y_2, \tilde{z}_f, \tilde{z}_b\right) < -\beta\right] \le \alpha \tag{10}$$

$$x_2 \ge -x_1 \tag{11}$$

$$y_2 \ge 0. \tag{12}$$

Inequality (9) is the second-stage budget constraint; the distributor can use the remaining budget from stage 1 in addition to the money generated through the sale of futures in stage 2 (when $x_2 < 0$). Inequality (10) is the second-stage VaR constraint; the distributor requires that the probability of loss more than β ($< B$) is no more than α. Alternatively said, the probability of realized profit less than $-\beta$ should not exceed α. Inequality (11) indicates that the distributor cannot sell more futures in stage 2 than the amount purchased in stage 1. For given x_1, y_1, w, m, we let (x_2^*, y_2^*) denote the optimal solution, i.e.,

$$\left(x_2^*\left(x_1, y_1, w, m,\right), y_2^*\left(x_1, y_1, w, m,\right)\right) = \arg\max_{x_2, y_2} E\left[\Pi\left(x_1, y_1, w, m, x_2, y_2, \tilde{z}_f, \tilde{z}_b\right)\right] \text{ s.t. (9) – (12).}$$

Let $z_{f\alpha}$ and $z_{b\alpha}$ denote the realizations of $\tilde{z}_f$ and $\tilde{z}_b$ at fractile α, i.e., $P\left[\tilde{z}_f \le z_{f\alpha}\right] = P\left[\tilde{z}_b \le z_{b\alpha}\right] = \alpha$. We assume that $z_{f\alpha} < 0$ and $z_{b\alpha} < 0$, i.e., the fractile parameter is such that the risk-averse decision maker in September of year t is concerned about profit realizations in May of year $t + 1$ that are below expectation. We also assume that the VaR constraint is satisfied in the event the distributor invests the entire budget in bottles, i.e.,

$$(1 - b_3(m_L) - z_{b\alpha})B < \beta. \tag{13}$$

This assumption is consistent with the practice of distributors who invest solely in bottled wine.

At the beginning of stage 1, the distributor selects x_1 and y_1 to maximize expected profit at the end of stage 2 subject to budget and VaR constraints:

$$\max_{x_1, y_1 \geq 0} E\left[\Pi\left(x_1, y_1, \tilde{w}, \tilde{m}, x_2^*\left(x_1, y_1, \tilde{w}, \tilde{m}\right), y_2^*\left(x_1, y_1, \tilde{w}, \tilde{m}\right), \tilde{z}_f, \tilde{z}_b\right)\right] \tag{14}$$

subject to

$$x_1 + y_1 \leq B \tag{15}$$

$$P\left[\Pi\left(x_1, y_1, w, m, x_2^*\left(x_1, y_1, w, m\right), y_2^*\left(x_1, y_1, w, m\right), \tilde{z}_f, \tilde{z}_b\right) < -\beta\right] \leq \alpha \text{ for all } (w, m) \in \Omega \tag{16}$$

Inequality (15) states that the distributor's initial investment in futures and bottles cannot exceed the allotted budget B. Inequality (16) is the VaR constraint under a time-consistent risk measure (e.g., see Boda and Filar 2006 or Devalkar et al. 2015). Some first-stage decisions (x_1, y_1) can satisfy the VaR constraint in stage 1 but may not comply with the VaR constraint in stage 2; such decisions lead to time-inconsistency and are not feasible in our model. To assure that risk aversion is time consistent over the planning horizon, the distributor must account for the VaR constraint in stage 2, and in particular, the choice of (x_1, y_1) must be such that there exists a solution to the stage-2 problem that satisfies the stage-2 VaR constraint for any realization (w, m) of $(\tilde{w}, \tilde{m})$.

We focus on understanding how investment in futures and bottles affect performance ceteris paribus, and therefore, we assume equal and positive expected returns at the end of stage 2, i.e.,

$$E[f_3(\tilde{w}, \tilde{m}) + \tilde{z}_f] = E[b_3(\tilde{m}) + \tilde{z}_b] > 1. \tag{17}$$

We relax this assumption in Section 4.

3.3. Analysis

We begin our analysis by partitioning the support Ω into three sets that identify realizations of $(\tilde{w}, \tilde{m})$ where the distributor would improve expected profit at the end of stage 2 by (1) selling futures, (2) buying futures, and (3) selling futures and buying bottles.

$\Omega0 = \{(w, m) \in \Omega : f_3(w, m)/f_2(w, m) = b_3(m)/b_2(m) = 1\}$

$\Omega1 = \{(w, m) \in \Omega : f_3(w, m)/f_2(w, m) < 1 \text{ and } b_3(m)/b_2(m) < 1\}$

$\Omega2 = \{(w, m) \in \Omega : f_3(w, m)/f_2(w, m) \geq \max\{b_3(m)/b_2(m), 1\} \setminus \Omega0\}$

$\Omega3 = \{(w, m) \in \Omega : b_3(m)/b_2(m) \geq \max\{f_3(w, m)]/f_2(w, m), 1\} \cup \Omega0\}$.

We define m_τ as $b_3(m_\tau)/b_2(m_\tau) = 1$ and $f_3(0, m_\tau)/f_2(0, m_\tau) = 1$, and $w_\tau(m)$ as $f_3(w_\tau(m), m)/f_2(w_\tau(m), m) = 1$ for $m \leq m_\tau$. Let $w_\tau^- = w_\tau(m_L)$. Note that

$$m_\tau < 0,\ w_\tau(m) < 0 \text{ for all } m < m_\tau, \text{ and } w_\tau(m_\tau) = 0 \tag{18}$$

(follows from (5), (6), (17)). In our analysis, we assume that

$$m_\tau > m_L \text{ and } w_\tau(m_L) > w_L. \tag{19}$$

Note that the set $\Omega1$ defines realizations where the expected return on futures and bottles over stage 2 is negative. A reversal of $m_\tau > m_L$ in (19) eliminates $\Omega1$, which is advantageous to any decision-maker regardless of whether she is risk-averse or risk-neutral. A reversal of $w_\tau(m_L) > w_L$ in (19) (while keeping $E[\tilde{w}] = 0$) implies a reduced weather uncertainty on behalf of wine futures, reducing the riskiness of this asset. As a consequence, (19) represents a riskier condition, and thus, our results remain intact when (19) does not hold. Figure 3 illustrates the above notation.

We make use of expressions that rely on the solution to the stage-2 problem with the VaR constraint (10) relaxed, which we denote as $\left(x_2^0, y_2^0\right)$, i.e.,

$$\left(x_2^0\left(x_1, y_1, w, m\right), y_2^0\left(x_1, y_1, w, m\right)\right) = \underset{x_2, y_2}{\arg\max}\, E\left[\Pi\left(x_1, y_1, w, m, x_2, y_2, \tilde{z}_f, \tilde{z}_b\right)\right] \text{ s.t. (9), (11), (12).}$$

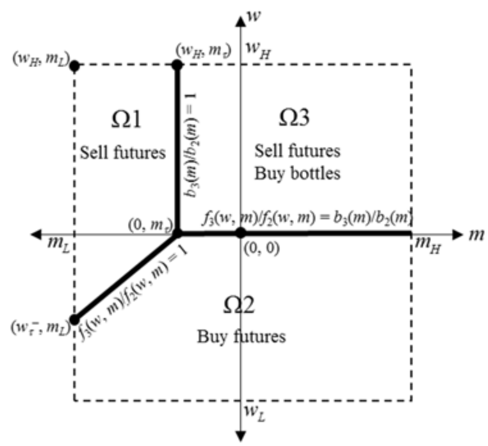

Figure 3. Illustration of sets Ω1 – Ω3. Function $w_\tau(m)$ is the line connecting points (w_τ^-, w_m) and (0, m_τ).

From the structure illustrated in Figure 3, it is clear that $\left(x_2^0, y_2^0\right)$ is given as follows:

$$\left(x_2^0, y_2^0\right) = \begin{cases} \left(-x_1, 0\right) & \text{if } \left(w, m\right) \in \Omega 1 \\ \left(\left(B - x_1 - y_1\right)/f_2\left(w, m\right), 0\right) & \text{if } \left(w, m\right) \in \Omega 2 \\ \left(-x_1, \left(B - x_1 - y_1 + f_2\left(w, m\right)x_1\right)/b_2\left(m\right)\right) & \text{if } \left(w, m\right) \in \Omega 3 \end{cases}, \tag{20}$$

(see Lemma A1 in the appendix for its derivation). Throughout our analysis we assume that, compared to no investment at the beginning of stage 1 (i.e., $x_1 = y_1 = 0$), an investment in some bottles increases expected profit:

$$\partial E\left[\Pi\left(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b\right)\right]/\partial y_1\Big|_{(x_1, y_1) = (0,0)} > 0. \tag{21}$$

In practice, (21) is likely to hold; otherwise, a distributor would not operate in this business. Inequality (21) implies that bottles command a higher expected return than holding cash in stage 1 as evidenced by purchases of bottles that occur each spring at the distributor motivating our study.

Proposition 1. *For any* (x_1, y_1),

$$\frac{\partial E\left[\Pi\left(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b\right)\right]}{\partial x_1} \geq \frac{\partial E\left[\Pi\left(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b\right)\right]}{\partial y_1} > 0. \tag{22}$$

Proposition 1 states that, at the beginning of stage 1 and for any current investment level, additional investment in futures is more profitable than additional investment in bottles for a risk-neutral distributor, and that both investment alternatives are more profitable than holding cash. The result hints that futures offer an inherent advantage over bottles. This advantage stems from the additional flexibilities of liquidity (i.e., being able to sell futures after observing weather and market random variables) and swapping (i.e., the ability to sell futures and buy bottles). The appendix provides the derivations and the resulting expressions for the valuation of liquidity, swapping, the combination of liquidity and swapping, as well as the value of holding cash in stage 1. Proposition A1 in the appendix shows how these values (i.e., liquidity, swapping, the combination of liquidity and swapping, and holding cash) change with increasing variance in weather and market random variables. It is important to indicate that while the values of liquidity and cash increase with higher variance in weather and market, the values of swapping and the combination of liquidity and swapping increase with only higher variation in market, and not necessarily with higher variation in weather. As a consequence of these observations, the next proposition establishes the impact of variance in weather and market on the expected profit function.

Proposition 2. *When $\phi_w(w)$ and $\phi_m(m)$ follow symmetric pdf, (a) $E\left[\Pi\left(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b\right)\right]$ increases in σ_m^2; (b) $E\left[\Pi\left(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b\right)\right]$ increases in σ_w^2 if the combined value from liquidity and swapping increases in σ_w^2.*

Proposition 2 shows that, for symmetric distributions, the expected profit increases in σ_m^2, however, it may increase or decrease in σ_w^2. Profit improvement from higher variation in market and weather uncertainty is enabled because of the recourse flexibility that allows the distributor to change its futures and bottles position based on the realization of the two random variables. When the value from the combination of liquidity and swapping increases in the variation in weather, then the expected profit also increases with higher degrees of weather uncertainty.

The preceding analysis has focused on the stage-1 profit function for a risk-neutral distributor. We build on this analysis in our derivation of the optimal solution to the risk-averse distributor problem defined in (8) – (16) in Proposition 3 below. The proposition makes use of the following notation and inequalities:

$x_1^+ = \beta/[1 - f_2(w_H, m_L)]$

$x_1^V = [\beta + z_{b\alpha} B]/([1 - f_2(w_H, m_\tau)][1 + z_{b\alpha}])$

$y_1^V = [\beta - [1 - f_2(w_H, m_L)]x_1^V]/[1 - b_3(m_L) - z_{b\alpha}]$

$x_1^s = (\beta - B[1 - b_3(m_L) - z_{b\alpha}])/[b_3(m_L) + z_{b\alpha} - f_2(w_H, m_L)]$

$y_1^s = (B[1 - f_2(w_H, m_L)] - \beta)/[b_3(m_L) + z_{b\alpha} - f_2(w_H, m_L)]$

$$- z_{f\alpha} < \beta/B \tag{23}$$

$$\frac{\partial E\left[\Pi\left(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b\right)\right] / \partial y_1\Big|_{(x_1,y_1)=(0,0)}}{\partial E\left[\Pi\left(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b\right)\right] / \partial x_1\Big|_{(x_1,y_1)=(0,0)}} < \frac{1 - b_3\left(m_L\right) - z_{b\alpha}}{1 - f_2\left(w_H, m_L\right)}. \tag{24}$$

The value of x_1^+ is the number of futures that cause constraint (16) to be binding (i.e., satisfied exactly) at point (w_H, m_L) given that $y_1 = 0$. The value of x_1^V is the number of futures that cause constraint (16) to be binding (i.e., satisfied exactly) at point (w_H, m_τ), which is independent of the value of y_1. The value of y_1^V is the number of bottles that cause constraint (16) to be binding (i.e., satisfied exactly) at point (w_H, m_L) given that $x_1 = x_1^V$. The values of x_1^s and y_1^s are the number of futures and bottles, respectively, that cause constraint (16) at point (w_H, m_L) to be intersecting with the budget constraint (15). The value of x_1^s is strictly smaller than x_1^+ when $x_1^+ < B$.

Inequality (23) restricts the variation in the randomness in futures at the end of stage 2. It implies that having the entire budget invested in futures in stage 2 at point (w_{τ}^-, m_L) does not violate the VaR constraint (10). Note that at point (w_{τ}^-, m_L), the risk-neutral distributor would keep all futures, and purchase additional futures if the budget allows. Inequality (23) is a rather mild condition. Recall (13), which says the VaR constraint is not violated if the distributor uses the entire budget to purchase bottles at the beginning of stage 1 (a condition supported by observed practice), i.e., $- z_{b\alpha} < \beta/B - [1 - b_3(m_L)] < \beta/B$. A comparison of (23) with (13) shows that our model allows for greater uncertainty in the randomness in futures prices than that in bottle prices. Unlike (13), inequality (23) does not mean that investing the entire budget in futures in stage 1 would not violate the VaR constraint (16). Rather, investing the entire budget in futures in stage 1 under (23) may violate the VaR constraint (16) at (w_H, m_τ) and (w_H, m_L).

Inequality (24) is used as a condition in characterizing the optimal solution. It compares the ratio of marginal returns from bottles to futures with the ratio of worst loss from bottles at α-fractile (i.e., $1 - b_3(m_L) - z_{b\alpha}$) to futures ($1 - f_2(w_H, m_L)$) because the distributor can liquidate futures at the worst weather and market realization (w_H, m_L). When (24) holds, the firm prefers futures more than bottles even at the worst realizations of weather and market random variables; when the opposite of (24) holds, the firm prefers bottles over futures.

The following proposition characterizes the optimal solution in both stages.

Proposition 3. *When* (23) *holds and* $\left(\tilde{z}_f, \tilde{z}_b\right)$ *follow a bivariate normal distribution*,

(*a*) *If* $\{x_1^+, x_1^V\} \geq B$, *then* $(x_1^*, y_1^*) = (B, 0)$ *and* $(x_2^*, y_2^*) = (x_2^0, y_2^0)$;

(b) *If* $x_1^V < B \leq x_1^+$, *then* $(x_1^*, y_1^*) = (x_1^V, B - x_1^V)$ *and* $(x_2^*, y_2^*) = (x_2^0, y_2^0)$;

(c) *If* $x_1^+ < \{x_1^V, B\}$, *then*

(i) *if* (24) *holds, then* $(x_1^*, y_1^*) = (x_1^+, 0)$ *and* $(x_2^*, y_2^*) = (x_2^0, y_2^0)$;

(ii) *if* (24) *does not hold*, *then* $(x_1^*, y_1^*) = (x_1^s, y_1^s)$ *and* $(x_2^*, y_2^*) = (x_2^0, y_2^0)$;

(d) *If* $x_1^s < x_1^V \leq x_1^+ < B$, *then*

(i) *if* (24) *holds, then* $(x_1^*, y_1^*) = (x_1^V, y_1^V)$ *and* $(x_2^*, y_2^*) = (x_2^0, y_2^0)$;

(ii) *if* (24) *does not hold*, *then* $(x_1^*, y_1^*) = (x_1^s, y_1^s)$ *and* $(x_2^*, y_2^*) = (x_2^0, y_2^0)$;

(e) *If* $x_1^V \leq x_1^s < x_1^+ < B$, *then* $(x_1^*, y_1^*) = (x_1^V, B - x_1^V)$ *and* $(x_2^*, y_2^*) = (x_2^0, y_2^0)$.

Proposition 3 leads to our main conclusion: It is always optimal to invest in at least some futures because $x_1^* > 0$ in all conditions (see the proof). While it is optimal to invest in futures, it is not necessarily to do so in bottles as in the conditions designated in Proposition 3(a) and 3(c)(i). This result holds true in spite of the additional uncertainty from weather that is present in futures which is not present in bottles. It should also be noted here that Propositions 3(a) and 3(c)(i) do not require that $\left(\tilde{z}_f, \tilde{z}_b\right)$ follow a bivariate Normal distribution.

The preceding analysis has built the second-stage results using the fact that the firm can invest its entire budget in futures in stage 2, i.e., when (23) holds. However, when (23) does not hold, the optimal second-stage decisions can be restricted by the VaR constraint (10); thus, x_2^* can be less than x_2^0. The next proposition shows that the firm should invest a positive amount of money in futures even if the second-stage decisions are limited by the VaR constraint (10).

Proposition 4. *When* $\phi_w(w)$ *follows a symmetric pdf and* $\left(\tilde{z}_f, \tilde{z}_b\right)$ *follow a bivariate normal distribution*,

$$\frac{\partial E\left[\Pi\left(x_1, y_1, \tilde{w}, \tilde{m}, x_2^*, y_2^*, \tilde{z}_f, \tilde{z}_b\right)\right]}{\partial x_1} \geq \frac{\partial E\left[\Pi\left(x_1, y_1, \tilde{w}, \tilde{m}, x_2^*, y_2^*, \tilde{z}_f, \tilde{z}_b\right)\right]}{\partial y_1} > 0. \quad (25)$$

In conclusion, combining the results of propositions 3 and 4, our analysis shows that the firm should always make a positive investment in wine futures despite the fact that they are tagged as the riskier asset when compared to bottled wine. This is a robust result because it holds under various general conditions, regardless of whether (23) holds or not.

4. Financial Benefits from Our Proposed Model

Our work is motivated by the world's largest wine distributor that does not invest in wine futures due to lack of knowledge about futures prices and their evolution to bottle prices. How significant is the economic benefit from investing in wine futures? This section demonstrates the financial benefits from using our model and trading futures compared with a benchmark of a distributor that trades only bottled wine. The appendix provides a detailed description of our data set (provided by Liv-ex) for the 44 leading Bordeaux winemakers used in our analysis.

We first calibrate our empirical models (models 1A, 1B, 2A, and 2B) to estimate the coefficients of weather and market variables for calendar year $t \in \{2008, 2009, 2010\}$. Using these coefficient estimates, we then solve the distributor's problem of allocating budget between the futures of vintage $t-1$ and the bottles of vintage $t-2$ for each winemaker $j \in \{1,\ldots, 44\}$ independently in May of calendar year $t \in \{2011, 2012\}$. Thus, the distributor plans her trading strategy for each winemaker independent of other winemakers.

In May of calendar year $t \in \{2011, 2012\}$, we assume that the distributor knows the distributions of all four random variables: $\tilde{w}$, $\tilde{m}$, $\tilde{z}_f$, and $\tilde{z}_b$. We use the five most-recent observations of weather and market random variables (w and m) in order to construct 25 equally likely scenarios for ($\tilde{w},\tilde{m}$), resulting in discrete uniform distributions, such that $E[\tilde{w}] = E[\tilde{m}] = 0$. Furthermore, we use the residuals from models 1B and 2B in order to construct the distributions of $\tilde{z}_f$ such that $E[\tilde{z}_f] = 0$ and $\tilde{z}_b$ such that $E[\tilde{z}_b]$ = 0, respectively. From these two distributions, we identify the α-fractile values corresponding to the values of $z_{f\alpha}$ and $z_{b\alpha}$ in our model.

In May of calendar year $t \in \{2011, 2012\}$, the distributor knows the actual futures and bottle prices (f_1 and b_1, respectively) for each winemaker from our data. Using the coefficient estimates from our empirical models, we then compute the prices in September of calendar year t (i.e., $f_2(w, m)$ and $b_2(m)$) and in May of calendar year $t + 1$ (i.e., $f_3(w, m) + z_f$ and $b_3(m) + z_b$) for given realizations of all four random variables.

We assume that the distributor's tolerable loss is 20% of budget (i.e., $\beta = 0.2B$), and we capture the effect of varying risk aversion by evaluating performance at $\alpha \in \{1, 0.20, 0.10\}$. The case of $\alpha = 1$ corresponds to a risk-neutral distributor, whereas $\alpha = 0.20$ and $\alpha = 0.10$ correspond to low risk-averse and high risk-averse distributors, respectively. We emphasize, however, that our results are independent of the choice of B, and we use $B = 10000$ in our numerical illustrations.

We denote $E[\Pi_I^{j,t}(x_1^*, y_1^*)]$ as the optimal profit coming from winemaker j who invests in futures and bottled wine in year t, and $E[\Pi_I^{j,t}(0, y_1^{**})]$ as the expected profit from the distributor's current practice of investing only in bottled wine with no investment in futures, i.e., $(x_1, x_2) = (0, 0)$. We define the financial benefit from using our model as follows:

$$\Delta^{j,t} = (E[\Pi_I^{j,t}(x_1^*, y_1^*)] - E[\Pi_I^{j,t}(0, y_1^{**})])/E[\Pi_I^{j,t}(0, y_1^{**})] \quad (26)$$

Table 3 summarizes the benefits from using our model of investing in futures, bottles and leaving cash under budget (equal for each winemaker) and VaR constraints described in (8) – (16). It presents the average benefit in this study as $\overline{\Delta}^{j} = (1/2)\sum_t(\Delta^{j,t})$ for each of the Bordeaux winemakers at different levels of risk aversion using tighter requirements regarding the probability of loss (α).

Winemaker (j)	Risk Neutral $\overline{\Delta}^{j}$	Low Risk Aversion $\overline{\Delta}^{j}$	High Risk Aversion $\overline{\Delta}^{j}$	Winemaker (j)	Risk Neutral $\overline{\Delta}^{j}$	Low Risk Aversion $\overline{\Delta}^{j}$	High Risk Aversion $\overline{\Delta}^{j}$
Angelus	4.45%	7.40%	10.00%	Lagrange St Julien	23.67%	23.67%	23.67%
Ausone	48.33%	53.18%	54.32%	Latour	70.13%	78.21%	78.84%
Beychevelle	0.00%	0.00%	0.00%	Leoville Barton	18.63%	18.63%	21.58%
Calon Segur	1.88%	1.88%	1.88%	Leoville Las Cases	28.20%	24.78%	25.92%
Carruades de Lafite	37.10%	51.70%	56.93%	Leoville Poyferre	36.72%	23.82%	23.39%
Cheval Blanc	29.71%	34.44%	36.89%	Lynch Bages	20.97%	20.97%	20.97%
Clos Fourtet	38.92%	38.96%	39.30%	Margaux	31.84%	50.52%	53.81%
Conseillante	10.69%	5.95%	5.35%	Mission Haut Brion	9.50%	12.99%	12.62%
Cos d'Estournel	36.04%	31.53%	31.99%	Montrose	14.90%	14.07%	17.98%
Ducru Beaucaillou	0.00%	2.30%	4.33%	Mouton Rothschild	10.93%	20.65%	22.62%
Duhart Milon	10.35%	8.94%	12.74%	Palmer	0.00%	0.00%	0.00%
Eglise Clinet	13.28%	21.90%	21.71%	Pavie	24.46%	25.99%	28.53%
Evangile	14.48%	33.16%	34.81%	Pavillon Rouge	5.00%	5.00%	5.00%
Figeac	84.78%	76.61%	74.73%	Petit Mouton	3.69%	3.69%	3.69%
Fleur Petrus	24.80%	30.97%	46.23%	Petrus	21.31%	17.63%	16.70%
Forts Latour	30.24%	30.24%	30.24%	Pichon Baron	17.06%	17.06%	17.06%
Grand Puy Lacoste	25.13%	26.18%	27.41%	Pichon Lalande	10.29%	5.85%	7.49%
Gruaud Larose	7.34%	7.34%	7.34%	Pin	5.00%	5.12%	6.04%
Haut Bailly	1.38%	1.38%	1.38%	Pontet Canet	10.44%	10.44%	10.44%
Haut Brion	9.91%	11.94%	14.32%	Talbot	0.00%	0.00%	0.00%
Lafite Rothschild	22.06%	43.32%	47.28%	Troplong Mondot	32.24%	31.29%	31.21%
Lafleur	55.74%	35.73%	33.29%	Vieux Chateau Certan	21.33%	29.73%	31.83%

	Risk Neutral $\overline{\Delta}$	**Low Risk Aversion** $\overline{\Delta}$	**High Risk Aversion** $\overline{\Delta}$
Average	**21.45%**	**22.98%**	**24.29%**

Table 3. The average financial benefit $\overline{\Delta} = \sum_j \overline{\Delta}^{j} \big/ 44$ where $\overline{\Delta}^{j}$ is the average profit improvement for winemaker j, B = 10000 and β = 2000; and, $\alpha \in \{1, 0.20, 0.10\}$ for risk neutral, low risk aversion, and high risk aversion, respectively.

These results show that even the largest distributors, which can be assumed to be risk neutral, would significantly benefit from investing in wine futures. The average expected profit improvement from these 44 Bordeaux wineries is 21.45% where the largest average improvement is observed at 84.78% at Figeac. In our numerical analysis, we relax the assumption that wine futures and bottled wine have equal expected returns as designated in (17). As a consequence of this relaxed constraint, the improvement from investing in wine futures can disappear when $E[f_3(\tilde{w},\tilde{m}) + \tilde{z}_f]/f_1$ is significantly smaller than $E[b_3(\tilde{m}) + \tilde{z}_b]/b_1$. Wine futures of four winemakers (e.g., Beychevelle) does not improve profits for the distributor.

Table 3 also demonstrates that our model leads to greater benefits in the presence of risk aversion. We observe that higher degrees of risk aversion increases the average profit improvement to 22.98% and 24.29%, respectively. In effect, the introduction of risk aversion on the benchmark case may force the distributor to hold excess cash, i.e., $y_1^{**} < B/b_1$. However, the flexibility of futures may lead to a greater total investment in stage 1 (i.e., $f_1x_1^* + b_1y_1^* > b_1y_1^{**}$) that translates into greater average improvement than that for a risk-neutral distributor where $f_1x_1^* + b_1y_1^* = b_1y_1^{**} = B$. This also indicates that relaxing (13) makes the benefits of our model even more profound. Therefore, we can conclude that our model advocating the trading of wine futures is generally more beneficial for risk-averse distributors. Though, risk aversion does not have a monotone impact, i.e., the average profit improvement can decrease for some winemakers (e.g. Conseillante) with higher risk aversion.

The financial benefits reported in Table 3 have significant implications for the wine industry as it complements the discussion regarding the need to establish a wine futures market in the US. Noparumpa et al. (2015) has shown that Bordeaux winemakers improve their profits by approximately 10% due to the wine futures market, and small and artisanal winemakers in the US can increase their profits by approximately 15%. Their study shows the positive effect through the use of tasting expert opinions. Table 3 shows that winemakers are not the only constituent benefiting from the wine futures market, and more importantly, wine distributors can benefit significantly when price evolutions can be predicted and a wine futures market is established in the US. In our finding, we utilize a different information, weather and market fluctuations, in demonstrating the financial benefits for distributors.

5. Conclusions

We have examined a wine distributor's problem that arises in May of every year, involving the selection between wine futures of the previous year's vintage and bottled wine made from grapes harvested two years ago.

Our paper makes three significant contributions. First, we develop an analytical model in order to determine the optimal selection of bottled wine and wine futures under weather and market uncertainty. The model is built on an empirical foundation where we explain the price evolution of futures and bottles based on the weather of the upcoming vintage and changes in market conditions. The analytical model employs the following information from the empirical analysis that uses a comprehensive data set regarding the trade of 44 most influential Bordeaux winemakers: (1) Futures price of a vintage is negatively influenced by a warmer growing season for the upcoming vintage, leading to a lower bottle price; (2) bottle prices are not influenced by weather conditions; and, (3) improving market conditions lead to increases in futures and bottle prices. We describe the market fluctuations through the changes in the Liv-ex 100 index. In this end, the identification of the Liv-ex 100 index as an explaining variable of the fluctuations in young wine prices also constitutes another contribution to the literature.

Second, we describe the optimal selection of bottled wine and wine futures with a limited budget and using a value-at-risk measure under weather and market uncertainty. We develop the structural properties of the optimal decisions. We conclude that a distributor should always invest in wine futures because it increases expected profit despite being a riskier asset than bottled wine.

Third, we demonstrate the financial benefits from using our analytical model through the numerical illustration using the same data for a large wine distributor. The hypothetical average profit improvement is significant, and is higher than 21% under the assumption of equal budget allotted for each winemaker. Moreover, the hypothetical average profit improvement becomes higher under risk aversion. Considering the wine distributor with a revenue of $11.4 Billion that motivated our study, our analysis constitutes a significant economic benefit from our proposed model.

In addition to these three main findings, we also demonstrate the impact of variation in weather and market uncertainty on the distributor's profitability. We show that higher variation in market uncertainty increases the expected profit, however, higher variation in weather can cause both an increase and a decrease in expected profit.

Our findings have significant implications for the wine industry as it is likely to encourage wine distributors to invest in wine futures with better information and expectation. Moreover, it is likely to increase the trading volume in the financial platform Liv-ex, resulting in even better information than what our sample provides.

While the motivation for our empirical and analytical work stems from the wine industry, our modeling perspective applies to a wide range of products and services. In the wine industry, the weather information for the upcoming vintage can be perceived as an information signal that causes a re-evaluation of the quality perception in the eyes of the consumers. There are various industries that have similar structures. In the technology industry, for example, the information regarding the release of new products often negatively influences the price of the current products. This is similar to the consequences of observing an improved weather condition during the growing season of the upcoming vintage. What is unique in our study, however, is that the upcoming vintage's weather information, when it is a relatively colder summer, can lead to an increase in the price of the current vintage. This kind of price increase cannot be observed in the technology industry through new information regarding the upcoming products. The increase in prices are only observed after a significant amount of time as in valuable antiques. However, the price increase in our study occurs without having to wait for a long period of time. Thus, the problem investigated here has unique features as it combines similar characteristics of information signaling from various industries for a single product and in a short span of time.

Our study has some limitations. Longer time series data can be used to test and enrich the price evolution of wine futures and bottled wine. Our study employs data only from the most popular Bordeaux winemakers and ignores fine wine producers from other regions. Our work also sheds light into future research directions. A longer time series data can help develop models that predict the price of wine futures and bottled wine. Such prediction models can help other parties, e.g. restaurateurs and investors who engage in the trade of wine. Our model can be expanded to consider other financing options such as debts and loans in order to increase the distributor's budget allocation. Our study, along with Noparumpa et al. (2015), lead to an elevated desire to establish a futures market in the US. Future research needs to address regulatory policies and legal requirements in order to arrive at an economically healthy futures market. Moreover, future research can examine the benefits of dynamically adjusting the distributor's budget each year in a multi-period setting.

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Appendix

Wine Analytics: Fine Wine Pricing and Selection under Weather and Market Uncertainty

Details of Empirical Foundation: Data Collection and Sample Selection

Wine price data is collected from Liv-ex (www.liv-ex.com), the world's largest database for fine wine prices. Our sample is composed of five vintages of wine futures (2007 to 2011) and five vintages of bottled wine (2006 to 2010) of 44 Bordeaux wines that aggregates the price data of 43,837 transactions (10,451 via wine futures) corresponding to a total trade volume of 520,133 bottles.

We refer to the Liv-ex Bordeaux 500 index (shortly, Liv-ex 500) when determining the wines to be examined. This index is composed of the 10 most recent bottled vintages of 50 leading Bordeaux wines. Among those 50 wines, sweet Sauternes wines (Yquem, Climens, Coutet, Suduiraut, and Rieussec) are excluded from the sample since their production process and timeline are different than the traditional Bordeaux wines. Another wine, Bahans/Clarence Haut Brion, is also excluded from the analysis due to missing price data. The final sample is composed of 44 of the 50 leading Bordeaux winemakers.

The weather information is gathered for the Merignac station from TuTiempo.net. Daily maximum temperatures are collected for each growing season (i.e., May 1 – August 31) for the years from 2006 to 2012. We then calculate the average growing season temperature for every year.

Market fluctuations are captured through the Live-ex Fine Wine 100 index (shortly, Liv-ex 100). The percentage change in Liv-ex 100 index over each growing season (i.e., May 1 – August 31) is obtained for the years from 2008 to 2012. The 100 most sought-after wines belong to older vintages than the vintages used in our sample, and therefore, there is no overlap of wines with our sample.

Can our Liv-ex 100 index can be replaced with another financial market variable? We find Liv-ex 100 to be a strong indicator that is distinct from traditional financial indices. This can be seen from the correlation coefficients between the Liv-ex 100 index and the three popular financial indicators during same time period with our data involving futures and bottle prices between 2007 and 2014: The correlation coefficient with the Standard & Poor 500 index is –0.03, with the Financial Times 100 index is 0.11, with the Dow Jones index is 0.04 whereas the correlation coefficients between these three financial indices range from 0.92 to 0.99. Thus, Liv-ex 100 is not an arbitrarily chosen market indicator.

Proofs and Derivations

Lemma A1. $\left(x_2^0\left(x_1,y_1,w,m\right),y_2^0\left(x_1,y_1,w,m\right)\right)=\underset{x_2,y_2}{\arg\max}\, E\left[\Pi\left(x_1,y_1,w,m,x_2,y_2,\tilde{z}_f,\tilde{z}_b\right)\right]$ *s.t.* (9), (11), (12);

$$\left(x_2^0\left(x_1,y_1,w,m\right),y_2^0\left(x_1,y_1,w,m\right)\right)=\begin{cases}\left(-x_1,0\right) & \text{if } \left(w,m\right)\in\Omega 1\\ \left(\left(B-x_1-y_1\right)/f_2\left(w,m\right),0\right) & \text{if } \left(w,m\right)\in\Omega 2\\ \left(-x_1,\left(B-x_1-y_1+f_2\left(w,m\right)x_1\right)/b_2\left(m\right)\right) & \text{if } \left(w,m\right)\in\Omega 3\end{cases}.$$

Proof of Lemma A1. The first derivatives of the stage-2 objective function (8) are

$$\partial E[\Pi\left(x_1,y_1,w,m,x_2,y_2,\tilde{z}_f,\tilde{z}_b\right)]/\partial x_2=f_3(w,m)-f_2(w,m) \tag{27}$$

$$\partial E[\Pi\left(x_1,y_1,w,m,x_2,y_2,\tilde{z}_f,\tilde{z}_b\right)]/\partial y_2=b_3(m)-b_2(m). \tag{28}$$

We see that the decision that maximizes expected profit simply depends on the relative profitability of futures and bottles for a given (w, m). In Ω1, both (27) and (28) are negative (neither futures nor bottles are profitable on expectation) which leads to $x_2^0=-x_1$ and $y_2^0=0$ due to (11) and (12). In Ω2, (27) is nonnegative and greater than (28) (futures are more profitable on expectation) which leads to $x_2^0=[B-x_1-y_1]/f_2(w,m)$ and $y_2^0=0$ due to (9) and (12). In Ω3, (28) is nonnegative and no smaller than (27) (bottles are more profitable on expectation) which leads to $x_2^0=-x_1$ and $y_2^0=[B+(f_2(w,m)-1)x_1-y_1]/b_2(m)$ due to (9) and (11). ❑

We denote the value created from the futures liquidation option with V_l. We first partition Ω3 into the following two sets: $\Omega3_A=\{(w,m): b_3(m)/b_2(m)\geq 1>f_3(w,m)/f_2(w,m)\}$, $\Omega3_B=\{(w,m): b_3(m)/b_2(m)\geq f_3(w,m)/f_2(w,m)\geq 1\}$. Futures do not provide a profitable return in $\Omega3_A$, and continue to be profitable but dominated by the returns from bottles in $\Omega 3_B$. The distributor would sell futures in sets $\Omega 1$ and $\Omega 3_A$ in order to avoid any further losses. The value created from liquidity is:

$$V_l = \iint_{\Omega 1 \cup \Omega 3_A} \left(f_2(w,m) - f_3(w,m) \right) \phi_w(w) \phi_m(m)\, dwdm \geq 0. \tag{29}$$

In set $\Omega 3_A$ futures are not profitable, and the distributor sells them and swaps them with bottles. In $\Omega 3_B$ the distributor also benefits from the ability to swap futures, even though they are still profitable We denote the value created from the swapping flexibility with V_s, can express it as follows:

$$V_s = \iint_{\Omega 3} \left(f_2(w,m) \frac{b_3(m)}{b_2(m)} - f_3(w,m) \right) \phi_w(w) \phi_m(m)\, dwdm \geq 0. \tag{30}$$

We next define the value gained from liquidation and swapping with $V_{l \cup s}$. The distributor benefits from both liquidating and swapping in set $\Omega 3_A$; discounting the double counting, we get:

$$V_{l \cup s} = V_l + V_s - \iint_{\Omega 3_A} \left(f_2(w,m) - f_3(w,m) \right) \phi_w(w) \phi_m(m)\, dwdm \geq 0. \tag{31}$$

The distributor can benefit from holding cash in stage 1. This money can be used to purchase futures in $\Omega 2$ and bottles in $\Omega 3$. The value from holding cash in stage 1, denoted V_c, can be described as:

$$V_c = \iint_{\Omega 2} \left(\frac{f_3(w,m)}{f_2(w,m)} - 1 \right) \phi_w(w)\phi_m(m) dwdm + \iint_{\Omega 3} \left(\frac{b_3(m)}{b_2(m)} - 1 \right) \phi_w(w)\phi_m(m) dwdm \geq 0. \tag{32}$$

Using this notation, we can open up the expressions that appear in Proposition 1 (see the proof of Proposition 1 for supporting detail):

$$\partial E\left[\Pi\left(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b \right) \right] \big/ \partial y_1 = E\left[b_3(\tilde{m}) + \tilde{z}_b \right] - 1 - V_c$$

$$\partial E\left[\Pi\left(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b \right) \right] \big/ \partial x_1 = E\left[f_3(\tilde{w}, \tilde{m}) + \tilde{z}_f \right] - 1 - V_c + V_{l \cup s} =$$

$$\partial E\left[\Pi\left(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b \right) \right] \big/ \partial y_1 + V_{l \cup s}.$$

Proof of Proposition 1. Using (x_2^0, y_2^0) (see Lemma A1), we have

$$\begin{aligned}
&\partial E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]/\partial y_1 = E[b_3(\tilde{m}) + \tilde{z}_b] - \iint_{\Omega 1} \phi_w(w)\phi_m(m) dwdm \\
&\quad - \iint_{\Omega 2} \left(f_3(w,m) / f_2(w,m) \right) \phi_w(w)\phi_m(m) dwdm - \iint_{\Omega 3} \left(b_3(m) / b_2(m) \right) \phi_w(w)\phi_m(m) dwdm \\
&\quad = E[b_3(\tilde{m}) + \tilde{z}_b] - 1 - V_c
\end{aligned} \tag{33}$$

which is nonnegative because both integrands are nonnegative by definitions of $\Omega 2$ and $\Omega 3$.

$$\begin{aligned}
&\partial E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]/\partial x_1 = E[f_3(\tilde{w}, \tilde{m}) + \tilde{z}_f] - \iint_{\Omega 1} \phi_w(w)\phi_m(m) dwdm \\
&- \iint_{\Omega 2} \left(f_3(w,m) / f_2(w,m) \right) \phi_w(w)\phi_m(m) dwdm - \iint_{\Omega 3} \left(b_3(m) / b_2(m) \right) \phi_w(w)\phi_m(m) dwdm \\
&+ \iint_{\Omega 1} \left(f_2(w,m) - f_3(w,m) \right) \phi_w(w) \phi_m(m)\, dwdm + \iint_{\Omega 3} \left(f_2(w,m) \frac{b_3(m)}{b_2(m)} - f_3(w,m) \right) \phi_w(w) \phi_m(m)\, dwdm \\
&\quad = E[f_3(\tilde{w}, \tilde{m}) + \tilde{z}_f] - 1 - V_c + V_{l \cup s}
\end{aligned} \tag{34}$$

which is nonnegative because both integrands are nonnegative by definitions of $\Omega 1$ and $\Omega 3$.

Note that $E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]$ is linear in x_1 and y_1. As a consequence, (33) is positive for any (x_1, y_1) following from (21). Moreover, following from (17),

$$\partial E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]/\partial x_1 - \partial E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]/\partial y_1 = V_{l \cup s}$$

which is nonnegative for any (x_1, y_1). ❑

We next examine the impact of increasing variation in uncertainty both in weather and market random variables (denoted σ_w^2 and σ_m^2, respectively) on an investment strategy in stage 1. Due to the linearity of the futures and bottle price functions, the expected prices $E\left[f_2\left(\tilde{w},\tilde{m}\right)\right]$, $E\left[f_3\left(\tilde{w},\tilde{m}\right)+\tilde{z}_f\right]$, $E\left[b_2\left(\tilde{m}\right)\right]$, and $E\left[b_3\left(\tilde{m}\right)+\tilde{z}_b\right]$ do not change with different values of σ_w^2 and σ_m^2. Moreover, in the absence of a recourse flexibility that enables a wine distributor to change her futures and bottle positions, the expected profit would not change with increasing values of σ_w^2 and σ_m^2. However, the values from liquidity, swapping, and combination flexibilities, and cash, denoted V_l, V_s, $V_{l\cup s}$, and V_c in (29) – (32) change with higher values of σ_w^2 and σ_m^2. Under symmetric pdfs for weather and market random variables, i.e., $\phi_w(w) = \phi_w(-w)$ with $w_H = -w_L$ and $\phi_m(m) = \phi_m(-m)$ with $m_H = -m_L$, the following proposition establishes their behavior with respect to σ_w^2 and σ_m^2.

Proposition A1. *When $\phi_w(w)$ and $\phi_m(m)$ follow symmetric pdf,* (*a*) *the value from liquidity V_l in* (29) *increases in σ_w^2 and σ_m^2*; (*b*) *the value from cash V_c in* (32) *increases in σ_w^2 and σ_m^2*; (*c*) *the value from swapping V_s in* (30) *increases in σ_m^2*; (*d*) *the value from the combination of liquidity and swapping $V_{l\cup s}$ in* (31) *increases in σ_m^2.*

Proof of Proposition A1. Recall that $E[f_3(w, m) + \tilde{z}_f] = f_3(w, m)$ and $E[b_3(m) + \tilde{z}_b] = b_3(m)$. The price evolution of futures is already described as $\partial f_3(w, m)/\partial w < \partial f_2(w, m)/\partial w < 0$ and $\partial f_3(w, m)/\partial m > \partial f_2(w, m)/\partial m > 0$, and bottles as $\partial b_3(m)/\partial m > \partial b_2(m)/\partial m > 0$.

(a) With higher values of σ_w^2 for a symmetric pdf for $\phi_w(w)$, regions Ω1 and $\Omega 3_A$ expand. Because $\partial f_3(w, m)/\partial w < \partial f_2(w, m)/\partial w < 0$, V_l in (29) would be adding increasing values of $f_2(w, m) - f_3(w, m)$ at each increment of w_H. Thus, V_l *in* (29) increases in σ_w^2. Similarly, with higher values of σ_m^2 for a symmetric pdf for $\phi_m(m)$, region Ω1 expands. Because $\partial f_3(w, m)/\partial m > \partial f_2(w, m)/\partial m > 0$, V_l in (29) would be adding increasing values of $f_2(w, m) - f_3(w, m)$ at each reduction in m_L. Thus, V_l *in* (29) increases in σ_m^2.

(b) Increasing σ_w^2 for a symmetric pdf for $\phi_w(w)$ implies expanding region Ω2 by reducing w_L where $f_3(w, m)/f_2(w, m) > 1$ by definition of the set. Because $\partial f_3(w, m)/\partial w < \partial f_2(w, m)/\partial w < 0$, we would be adding increasing values of $[(f_3(w, m)/f_2(w, m)) - 1]$. Similarly, increasing σ_w^2 for a symmetric pdf for $\phi_w(w)$ implies expanding region Ω3 by increasing w_H where $b_3(m)/b_2(m) > 1$ by definition of the set. Because $\partial b_3(m)/\partial w = \partial b_2(m)/\partial w = 0$ and we would not be changing the second term of V_c in (32). The changes region Ω2 is positive, and therefore, V_c in (32) increases in σ_w^2. A similar proof follows for the impact of σ_m^2. Increasing σ_m^2 for a symmetric pdf for $\phi_m(m)$ implies expanding region Ω2 by reducing m_L and increasing m_H where $f_3(w, m)/f_2(w, m) > 1$ by definition of the set. Because $\partial f_3(w, m)/\partial m > \partial f_2(w, m)/\partial m > 0$, we would be adding increasing values of $[(f_3(w, m)/f_2(w, m)) - 1]$. Similarly, increasing σ_m^2 for a symmetric pdf for $\phi_m(m)$ implies expanding region Ω3 by increasing m_H where $b_3(m)/b_2(m) > 1$ by definition of the set. Because $\partial b_3(m)/\partial m > \partial b_2(m)/\partial m > 0$, we would be adding increasing values of $[(b_3(m)/b_2(m)) - 1]$ to the second term of V_c in (32). The changes in region Ω2 and Ω3 are positive, and therefore, V_c in (32) increases in σ_m^2.

(c) The value from swapping V_s in (30) is defined in Ω3. Increasing σ_m^2 for a symmetric pdf for $\phi_m(m)$ implies expanding Ω1 (by reducing m_L) and Ω3 (by increasing m_H). In Ω3, $b_3(m)/b_2(m) > 1$, and its value is increasing due to $\partial b_3(m)/\partial m > \partial b_2(m)/\partial m > 0$. At the new market realization greater than m_H, we know that $f_2(w, m)[b_3(m)/b_2(m)] - f_3(w, m) > 0$ because of the definition of Ω3 (so that the firm swaps futures with a more profitable bottle investment). Thus, expanding the support beyond m_H adds value and expanding the lower support below m_L does not cause any loss; therefore, V_s in (30) is increasing in σ_m^2.

(d) The proof follows from the proofs of parts (a) and (c). ❑

Proof of Proposition 2. (a) Increasing σ_m^2 for a symmetric pdf for $\phi_m(m)$ implies reducing m_L and increasing m_H. Reducing m_L to $m_L - \varepsilon$ (where $\varepsilon > 0$) and increasing m_H to $m_H + \varepsilon$ leads to three cases for investigation. Case 1: $(w, m_L - \varepsilon) \in$ Ω1 and $(w, m_H + \varepsilon) \in$ Ω3: Because $\partial f_3(w, m)/\partial m > \partial f_2(w, m)/\partial m > 0$ and because bottles are even more profitable than futures in Ω3, the losses from the futures investment at $(w, m_L - \varepsilon) \in$ Ω1 are smaller in absolute value than the gains $(w, m_H + \varepsilon) \in$ Ω3, and thus, the expected profit increases. Case 2: $(w, m_L - \varepsilon) \in$ Ω1 and $(w, m_H + \varepsilon) \in$ Ω2: If $(w, m_L - \varepsilon) \in$ Ω1, then because $\partial f_3(w,$

$m)/\partial m > \partial f_2(w, m)/\partial m > 0$, the losses from the futures investment at $(w, m_L - \varepsilon) \in \Omega 1$ are smaller in absolute value than the gains $(w, m_H + \varepsilon) \in \Omega 2$, and thus, the expected profit increases. Case 3: $(w, m_L - \varepsilon) \in \Omega 2$ and $(w, m_H + \varepsilon) \in \Omega 2$: If $(w, m_L - \varepsilon) \in \Omega 2$, the losses from the futures investment at $(w, m_L - \varepsilon) \in \Omega 2$ are recovered by the gains at $(w, m_H + \varepsilon) \in \Omega 2$ due to symmetry, and thus, the expected profit does not change. Combining the results from these three cases, the expected profit increases with higher levels of σ_m^2. (b) Using the proof of Proposition 1, the expected profit for any (x_1, y_1) pair is

$$E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)] = [E[f_3(\tilde{w}, \tilde{m}) + \tilde{z}_f] - 1 - V_c + V_{l\cup s}]x_1 + [E[b_3(\tilde{m}) + \tilde{z}_b] - 1 - V_c]y_1 + B\, V_c$$

$$= [E[f_3(\tilde{w}, \tilde{m}) + \tilde{z}_f] - 1 + V_{l\cup s}]x_1 + [E[b_3(\tilde{m}) + \tilde{z}_b] - 1]y_1 + (B - x_1 - y_1)V_c.$$

Increasing σ_w^2 does not change $E[f_3(\tilde{w}, \tilde{m}) + \tilde{z}_f]$ and $E[b_3(\tilde{m}) + \tilde{z}_b]$. Proposition A1(b) has shown that V_c is increasing in σ_w^2. Thus, it is sufficient to observe that $\partial E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]/\partial \sigma_w^2 > 0$ if the combined value from liquidity and swapping increases in σ_w^2, i.e., $\partial V_{l\cup s}/\partial \sigma_w^2 > 0$. ❑

Lemma A2. $[\partial E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]/\partial y_1 |_{(x1, y1) = (0,0)}]/[\partial E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]/\partial x_1 |_{(x1, y1) = (0,0)}]$ *equals to* $[\partial E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]/\partial y_1]/[\partial E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]/\partial x_1]$ *for any* (x_1, y_1).

Proof of Lemma A2. Follows from the linearity of $E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]$ in x_1 and y_1, as shown in the proof of proposition 1. ❑

Development of the proof of Proposition 3

We first define the following boundary sets: $\Omega 2^E = \{(w, m) \in \Omega 2: m < m_\tau, w = w_\tau(m)\}$ and $\Omega 3^E = \{(w, m) \in \Omega 3: m = m_\tau\}$. In the following analysis we examine the value of profit function $\Pi(x_1, y_1, w, m, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha})$ at three points, and use this analysis in the proof of Proposition 3. The three points identified in Figure A1 correspond the realizations of $(\tilde{w}, \tilde{m})$ that yields low values of $\Pi(x_1, y_1, w, m, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha})$.

Lemma A3. *If* (23), *then* $\Pi(x_1, y_1, w_\tau^-, m_L, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) \geq -\beta$ *for any* (x_1, y_1).

Proof of Lemma A3. Note that $(w_\tau^-, m_L) \in \Omega 2^E$. This implies $f_3(w_\tau^-, m_L)/f_2(w_\tau^-, m_L) = 1$ by definition of set. Thus, the realized profit at $(z_{f\alpha}, z_{b\alpha})$ is

$$\Pi(x_1, y_1, w_\tau^-, m_L, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) = [b_3(m_L) + z_{b\alpha} - 1]y_1 + z_{f\alpha}[B - y_1]. \tag{35}$$

Note first that (35) independent of x_1; because $f_3(w_\tau^-, m_L)/f_2(w_\tau^-, m_L) = 1$ and (5) imply that $f_3(w_\tau^-, m_L) = f_2(w_\tau^-, m_L) = 1$. Because $m_L < m_\tau$, it follows that $b_3(m_L)/b_2(m_L) < 1$, and thus from (6) it follows that $b_3(m_L) < b_2(m_L) < 1$. Combined with $z_{b\alpha} < 0$ (by assumption), they imply $b_3(m_L) + z_{b\alpha} - 1 < 0$. Following from (13), we have $[b_3(m_L) + z_{b\alpha} - 1]y_1 > -\beta$ for any $0 \leq y_1 \leq B$. Furthermore, following from (23), we have $z_{f\alpha}[B - y_1] > -\beta$ for any $0 \leq y_1 \leq B$. ❑

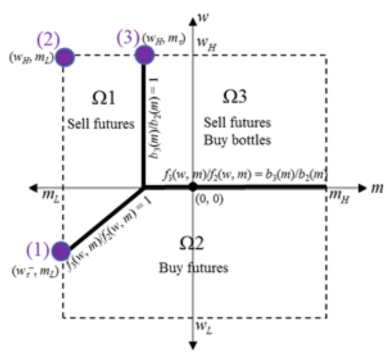

Figure A1. Points (1) – (3) are candidates for the minimum value of $\Pi(x_1, y_1, w, m, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha})$.

Lemma A4. *If* (23), *then* $\Pi(x_1, y_1, w, m, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) \geq -\beta$ *for all* $(w, m) \in \Omega 2$ *for any* (x_1, y_1).

Proof of Lemma A4. We first focus on $(w, m) \in \Omega 2^E$, for which $f_3(w, m)/f_2(w, m) = 1$, which in turn implies $f_3(w, m) = f_2(w, m) = 1$ for all $(w, m) \in \Omega 2^E$ (see (5)). Thus, for any $(w, m) \in \Omega 2^E$,

$\Pi(x_1, y_1, w, m, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha} \mid (w, m) \in \Omega2^E) = [b_3(m) + z_{b\alpha} - 1]y_1 + z_{f\alpha}[B - y_1] \geq [b_3(m_L) + z_{b\alpha} - 1]y_1 + z_{f\alpha}[B - y_1] = \Pi(x_1, y_1, w_\tau^-, m_L, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) \geq -\beta$ where the first inequality follows from $b_3(m)$ increasing in m, and the last inequality follows from Lemma A3.

Note that the expression above is independent of x_1 because $f_3(w, m) = f_2(w, m) = 1$ for all $(w, m) \in \Omega2^E$. For any $(w, m) \in \Omega2\backslash\Omega2^E$, we have $f_3(w, m)/f_2(w, m) > 1$ (by the definition of $\Omega2$). This implies that $f_3(w, m) > f_2(w, m) > 1$ (see (5)). Hence, the realized profit $\Pi(x_1, y_1, w, m, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha})$ is increasing in x_1 for any $(w, m) \in \Omega2\backslash\Omega2^E$, and thus $\Pi(x_1, y_1, w, m, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) \geq -\beta$ for all $(w, m) \in \Omega2$. ❑

Note that the profit at point $(w_H, m_L) \in \Omega1$ is

$$\Pi(x_1, y_1, w_H, m_L, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) = [f_2(w_H, m_L) - 1]x_1 + [b_3(m_L) + z_{b\alpha} - 1]y_1. \tag{36}$$

We define $x_1^H(y_1)$ which satisfies $\Pi_1(x_1^H(y_1), y_1, w_H, m_L, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) = -\beta$ for a given y_1, i.e.,

$$x_1^H(y_1) = [\beta - [1 - b_3(m_L) - z_{b\alpha}]y_1]/[1 - f_2(w_H, m_L)]. \tag{37}$$

Lemma A5. $\Pi(x_1, y_1, w_H, m_L, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) \geq -\beta$ *for any* $y_1 \leq B$ *and* $x_1 \leq x_1^H(y_1)$.

Proof of Lemma A5. We know that $f_2(w_H, m_L) < 1$ and $b_3(m_L) < 1$ (follows from $(w_H, m_L) \in \Omega1$, (5), and (6)). Also, $z_{b\alpha} < 0$ by assumption. Therefore, $\Pi(x_1, y_1, w_H, m_L, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha})$ in (36) is decreasing in x_1 and y_1. This also implies that $x_1^H(y_1)$ in (37) is decreasing in y_1. For any $y_1 \leq B$ (due to (13)) and $x_1 \leq x_1^H(y_1)$, $\Pi(x_1, y_1, w_H, m_L, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) \geq \Pi(x_1^H(y_1), y_1, w_H, m_L, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) = -\beta$. ❑

Lemma A6. $\Pi(x_1, y_1, w, m, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) \geq -\beta$ *for all* $(w, m) \in \Omega1$ *for any* $y_1 \leq B$ *and* $x_1 \leq x_1^H(y_1)$.

Proof of Lemma A6. Since $f_2(w, m)$ and $b_3(m)$ are increasing in m, and $f_2(w, m)$ is decreasing in w, $\Pi(x_1, y_1, w, m, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha} \mid (w, m) \in \Omega1) = [f_2(w, m) - 1]x_1 + [b_3(m) + z_{b\alpha} - 1]y_1 \geq [f_2(w_H, m_L) - 1]x_1 + [b_3(m_L) + z_{b\alpha} - 1]y_1 = \Pi(x_1, y_1, w_H, m_L, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) \geq -\beta$ where the last inequality follows from Lemma A5. ❑

Lemma A7. $\Pi(x_1, y_1, w_H, m_\tau, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) \geq -\beta$ *for any* $x_1 \leq x_1^V$.

Proof of Lemma A7. Note that $(w_H, m_\tau) \in \Omega3^E$ implies $b_3(m_\tau)/b_2(m_\tau) = 1$. This further implies $b_3(m_\tau) = b_2(m_\tau) = 1$ (due to (6)). Thus,

$$\Pi(x_1, y_1, w_H, m_\tau, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) = [f_2(w_H, m_\tau) - 1][1 + z_{b\alpha}]x_1 + z_{b\alpha}B. \tag{38}$$

Note that the expression above is independent of y_1. It is decreasing in x_1 for two reasons: First, $(w_H, m_\tau) \notin \Omega2$ implies that $f_3(w_H, m_\tau)/f_2(w_H, m_\tau) < 1$ which further implies $f_3(w_H, m_\tau) < f_2(w_H, m_\tau) < 1$ (due to (5)), and second, $1 + z_{b\alpha} > 0$ (due to (13) and $\beta < B$).

We define x_1^V which satisfies $\Pi(x_1^V, y_1, w_H, m_\tau, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) = -\beta$ for any y_1, i.e.,

$$x_1^V = [\beta + z_{b\alpha} B]/([1 - f_2(w_H, m_\tau)][1 + z_{b\alpha}]). \tag{39}$$

Therefore, $\Pi(x_1, y_1, w_H, m_\tau, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) \geq \Pi(x_1^V, y_1, w_H, m_\tau, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) = -\beta$ for any $x_1 \leq x_1^V$. ❑

Lemma A8. $\Pi(x_1, y_1, w, m, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) \geq -\beta$ *for all* $(w, m) \in \Omega3$ *for any* $x_1 \leq x_1^V$.

Proof of Lemma A8. We first focus on $(w, m_\tau) \in \Omega3^E$, for which $b_3(m_\tau) = b_2(m_\tau) = 1$ (follows from the definition of $\Omega3^E$ and (6)). The realized profit can be expressed as $\Pi(x_1, y_1, w, m, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha} \mid (w, m) \in \Omega3^E) = [f_2(w, m_\tau) - 1][1 + z_{b\alpha}]x_1 + z_{b\alpha}B \geq [f_2(w_H, m_\tau) - 1][1 + z_{b\alpha}]x_1 + z_{b\alpha}B = \Pi(x_1, y_1, w_H, m_\tau, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) \geq -\beta$ where the first inequality follows from $f_2(w, m_\tau)$ decreasing in w, and the last inequality follows from Lemma A7.

Note that the expression above is independent of y_1 because $b_3(m_\tau) = b_2(m_\tau) = 1$ for all $(w, m_\tau) \in \Omega3^E$. For any $(w, m) \in \Omega3\backslash\Omega3^E$, $b_3(m)/b_2(m) > 1$ by the definition of $\Omega3$. This further implies that $b_3(m) > b_2(m) > 1$ (due to (6)). Hence, the realized profit at $(z_{f\alpha}, z_{b\alpha})$ increases in y_1 for any $(w, m) \in \Omega3\backslash\Omega3^E$. Therefore,

$$\Pi(x_1, y_1, w, m, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha} \mid (w, m) \in \Omega3\backslash\Omega3^E) \geq -\beta.$$ ❑

Lemma A9. *Suppose that* (23) *holds. Then* $\Pi(x_1, y_1, w, m, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) \geq -\beta$ *for all* $(w, m) \in \Omega$ *for any* $y_1 \leq B$ *and* $x_1 \leq \min\{x_1^H(y_1), x_1^V\}$.

Proof of Lemma A9. Follows from lemmas A4, A6, and A8. ❑

Lemma A10. *Suppose that* (23) *holds. Then* $P[\Pi(x_1, 0, w, m, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b) < -\beta] \leq \alpha$ *for all* $(w, m) \in \Omega$ *for any* $x_1 \leq \min\{x_1^H(0), x_1^V\}$. *This means that* (x_2^0, y_2^0) *and* $(x_1, 0)$ *decisions such that* $x_1 \leq \min\{x_1^H(0), x_1^V\}$ *satisfy both* (10) *and* (16).

Proof of Lemma A10. $\Pi(x_1, 0, w, m, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b \mid (w, m) \in \Omega 1)$ has neither $\tilde{z}_f$ nor $\tilde{z}_b$ term. $\Pi(x_1, 0, w, m, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b \mid (w, m) \in \Omega 2)$ has only $\tilde{z}_f$, and $\Pi(x_1, 0, w, m, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b \mid (w, m) \in \Omega 3)$ has only $\tilde{z}_b$. We also know from Lemma A9 that $\Pi(x_1, 0, w, m, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) \geq -\beta$ for all $(w, m) \in \Omega$ for any $x_1 \leq \min\{x_1^H(0), x_1^V\}$ when $y_1 = 0$. Combined with $P\left[\tilde{z}_f \leq z_{f\alpha}\right] = P\left[\tilde{z}_b \leq z_{b\alpha}\right] = \alpha$, they imply that $P[\Pi(x_1, 0, w, m, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b) < -\beta] \leq \alpha$ for all $(w, m) \in \Omega$ for any $x_1 \leq \min\{x_1^H(0), x_1^V\}$. As a consequence, VaR constraints (10) and (16) are satisfied by (x_2^0, y_2^0) and $(x_1, 0)$ decisions for $x_1 \leq \min\{x_1^H(0), x_1^V\}$. ❑

Lemma A11. *Suppose that* (23) *holds*, *and* $\left(\tilde{z}_f, \tilde{z}_b\right)$ *follow a bivariate normal distribution*. *Then* $P[\Pi(x_1, y_1, w, m, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b) < -\beta] \leq \alpha$ *for all* $(w, m) \in \Omega$ *for any* $0 < y_1 < B$ *and* $x_1 \leq \min\{x_1^H(y_1), x_1^V\}$. *This means that* (x_2^0, y_2^0) *and* (x_1, y_1) *decisions such that* $0 < y_1 < B$ *and* $x_1 \leq \min\{x_1^H(y_1), x_1^V\}$ *satisfy both* (10) *and* (16).

Proof of Lemma A11. Note first that $y_1 \neq 0$. $\Pi(x_1, y_1, w, m, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b \mid (w, m) \in \Omega 1 \cup \Omega 3)$ has only $\tilde{z}_b$ term. Combined with $P\left[\tilde{z}_b \leq z_{b\alpha}\right] = \alpha$, and lemmas A6 and A8, it follows that $P[\Pi(x_1, y_1, w, m, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b) < -\beta] \leq \alpha$ for all $(w, m) \in \Omega 1 \cup \Omega 3$ for any $0 < y_1 < B$ and $x_1 \leq \min\{x_1^H(y_1), x_1^V\}$.

$\Pi(x_1, y_1, w, m, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b \mid (w, m) \in \Omega 2)$ has both $\tilde{z}_f$ and $\tilde{z}_b$ terms. We first consider the case where $\tilde{z}_f$ and $\tilde{z}_b$ are perfectly positively correlated, i.e., $\tilde{z}_f = k\,\tilde{z}_b$ where $k > 0$. This implies

$$P[\tilde{z}_f \leq z_{f\alpha} \;\&\; \tilde{z}_b \leq z_{b\alpha}] = P[k\,\tilde{z}_b \leq kz_{b\alpha} \;\&\; \tilde{z}_b \leq z_{b\alpha}] = P[\tilde{z}_b \leq z_{b\alpha}] = \alpha.$$

Together with Lemma A4, it follows that

$$P[\Pi(x_1, y_1, w, m, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b) < -\beta] \leq \alpha \tag{40}$$

for all $(w, m) \in \Omega 2$ for any (x_1, y_1). We then consider the less-than-perfect positive correlation case where $(\tilde{z}_f, \tilde{z}_b)$ follow a bivariate normal distribution. The randomness in profit can be expressed as

$$\tilde{Z}_\rho = (x_1 + x_2^0)\,\tilde{z}_f + (y_1 + y_2^0)\,\tilde{z}_b$$

where ρ is the correlation coefficient for $(\tilde{z}_f, \tilde{z}_b)$. As a consequence of bivariate normal distribution, $\tilde{Z}_\rho$, which is the sum of normal random variables, is a normal random variable with

$$E[\tilde{Z}_\rho] = 0 \text{ and } V\left[\tilde{Z}_\rho\right] = \left(x_1 + x_2^0\right)^2 \sigma_{\tilde{z}_f}^2 + \left(y_1 + y_2^0\right)^2 \sigma_{\tilde{z}_b}^2 + 2\rho\sigma_{\tilde{z}_f}\sigma_{\tilde{z}_b}.$$

From $E[\tilde{z}_f] = E[\tilde{z}_b] = 0$ and $\{z_{f\alpha}, z_{b\alpha}\} < 0$, it follows that $\alpha \leq 0.5$. Therefore,

$P[\Pi(x_1, y_1, w, m, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b) < -\beta] = P[\tilde{Z}_\rho < -\beta - \Pi(x_1, y_1, w, m, x_2^0, y_2^0, 0, 0)] \leq P[\tilde{Z}_1 < -\beta - \Pi(x_1, y_1, w, m, x_2^0, y_2^0, 0, 0)] \leq \alpha$ for all $(w, m) \in \Omega 2$ for any (x_1, y_1). The first inequality follows from $\alpha \leq 0.5$ and the fact that variance is increasing in ρ. The second inequality follows from (40), i.e., the case of perfect positive correlation. As a consequence, VaR constraints (10) and (16) are satisfied by (x_2^0, y_2^0) and (x_1, y_1) decisions such that $0 < y_1 < B$ and $x_1 \leq \min\{x_1^H(y_1), x_1^V\}$. ❑

Proof of Proposition 3. We begin with relaxing (10), i.e., $(x_2, y_2) = (x_2^0, y_2^0)$ is feasible. We then show that, when (23) holds, constraint (10) is nonbinding at the optimal solution to the problem defined in (8) – (16). From Proposition 1, we know that $(x_1, y_1) = (0, 0)$ cannot be optimal. Moreover, $x_1^+ = x_1^H(0) > 0$ (see (37)) due to $\beta > 0$ and $1 > f_2(w_H, m_L)$ (follows from $(w_H, m_L) \in \Omega 1$ and (5)).

Part (a): When $B \leq \min\{x_1^+, x_1^V\}$, then $(x_2, y_2) = (x_2^0, y_2^0)$ and $(x_1, y_1) = (B, 0)$ satisfy both (10) and (16) following from Lemma A10. This implies that $(x_2^*, y_2^*) = (x_2^0, y_2^0)$ by definition of (x_2^0, y_2^0). It follows from Proposition 1 that $(x_1^*, y_1^*) = (B, 0)$. Part (b): Note that $x_1^V < x_1^H(B - x_1^V)$ when $x_1^V < B \leq x_1^+$. Proposition 1 and Lemma A11 imply that $(x_2^*, y_2^*) = (x_2^0, y_2^0)$ and $(x_1^*, y_1^*) = (x_1^V, B - x_1^V)$.

Part (c): Note that $x_1^H(y_1)$ is linearly decreasing in y_1 (see (37)). As a consequence, when $x_1^+ < x_1^V$, we have $x_1^H(y_1) < x_1^V$ for any $y_1 \geq 0$. Moreover, $E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]$ is linear in x_1 and y_1 (see proof of Proposition 1). Therefore,

$$dE[\Pi(x_1^H(y_1), y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]/dy_1 = \partial E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]/\partial y_1 - \left[\frac{1-b_3(m_L)-z_{b\alpha}}{1-f_2(w_H, m_L)}\right]\partial E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]/\partial x_1.$$

Part (c)(i): $dE[\Pi(x_1^H(y_1), y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]/dy_1 < 0$ due to (24) and Lemma A2. Following from Lemma A10, $(x_2, y_2) = (x_2^0, y_2^0)$ and $(x_1, y_1) = (x_1^+, 0)$ satisfy both (10) and (16). Moreover, (15) is satisfied due to $x_1^+ < B$. Therefore, together with Proposition 1, it follows that $(x_2^*, y_2^*) = (x_2^0, y_2^0)$ and $(x_1^*, y_1^*) = (x_1^+, 0)$. Part (c)(ii): $dE[\Pi(x_1^H(y_1), y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]/dy_1 \geq 0$ due to the reversal of (24), and Lemma A2. Note that $x_1^H(B) > 0$ (see (13) and (37)). Together with $x_1^+ < B$ and the linearity of $x_1^H(y_1)$ in y_1, it follows that the VaR constraint (16) at (w_H, m_L) crosses the budget constraint at a single point, i.e., $y_1^s + x_1^H(y_1^s) = B$

such that $y_1^s = \dfrac{B[1-f_2(w_H, m_L)]-\beta}{[b_3(m_L)+z_{b\alpha}-f_2(w_H, m_L)]}$ and $x_1^s = x_1^H(y_1^s) = \dfrac{\beta - B[1-b_3(m_L)-z_{b\alpha}]}{[b_3(m_L)+z_{b\alpha}-f_2(w_H, m_L)]}$

where $\{x_1^s, y_1^s\} > 0$ following from $x_1^+ < B$ and (13). Note also that $x_1^s < x_1^+$. Following from Lemma A11, $(x_2, y_2) = (x_2^0, y_2^0)$ and $(x_1, y_1) = (x_1^s, y_1^s)$ satisfy both (10) and (16). Therefore, together with Proposition 1, it follows that $(x_2^*, y_2^*) = (x_2^0, y_2^0)$ and $(x_1^*, y_1^*) = (x_1^s, y_1^s)$. Part (d): We now examine the case when $x_1^s < x_1^V \leq x_1^+ < B$. Part (d)(i): When $x_1^V = x_1^+$, it follows from the proof of part (c)(i). When $x_1^V < x_1^+$, $x_1^H(y_1)$ linearly decreasing in y_1 implies that there exists a single y_1^V, i.e., $x_1^H(y_1^V) = x_1^V$ such that

$$y_1^V = \frac{\beta - \dfrac{[\beta + z_{b\alpha}B]}{[1+z_{b\alpha}]}\dfrac{[1-f_2(w_H, m_L)]}{[1-f_2(w_H, m_\tau)]}}{[1-b_3(m_L)-z_{b\alpha}]}$$

where $x_1^V + y_1^V < B$ (i.e., (15) is satisfied) due to $x_1^s < x_1^V$, $x_1^+ < B$, and (13). $dE[\Pi(x_1^H(y_1), y_1, \tilde{w}, \tilde{m}, x_2^0, y_2^0, \tilde{z}_f, \tilde{z}_b)]/dy_1 < 0$ due to (24) and Lemma A2. Together with Proposition 1 and Lemma A11, it follows that $(x_2^*, y_2^*) = (x_2^0, y_2^0)$ and $(x_1^*, y_1^*) = (x_1^V, y_1^V)$. Part (d)(ii): Since $x_1^s < x_1^V$, it follows from the proof of part (c)(ii). Part (e): Note that $x_1^V \leq x_1^H(B - x_1^V)$ when $x_1^V \leq x_1^s$. Proposition 1 and Lemma A11 imply that $(x_2^*, y_2^*) = (x_2^0, y_2^0)$ and $(x_1^*, y_1^*) = (x_1^V, B - x_1^V)$. ❑

Proof of Proposition 4. Relaxing (23) does not affect the feasibility of (x_2^0, y_2^0) in $\Omega 1$ and $\Omega 3$ (see lemmas A6 and A8). However, (x_2^0, y_2^0) may no longer be feasible in $\Omega 2$ (see Lemma A4). From (7), the realized profit at α-fractile is $\prod(x_1, y_1, w, m, x_2, y_2, z_{f\alpha}, z_{b\alpha}) = -x_1 - y_1 - f_2(w, m)x_2 - b_2(m)y_2 + [f_3(w, m) + z_{f\alpha}](x_1 + x_2) + [b_3(m) + z_{b\alpha}](y_1 + y_2)$ which is linear in $z_{f\alpha}$. Following from (11), $\partial\prod(x_1, y_1, w, m, x_2, y_2, z_{f\alpha}, z_{b\alpha})/\partial z_{f\alpha} \geq 0$. Therefore, it is sufficient to show that Proposition 4 holds at the extreme case such that $z_{f\alpha} \to -\infty$. The result naturally extends to any other $z_{f\alpha}$, which may or may not satisfy (23).

$z_{f\alpha} \to -\infty$ implies that $x_2^* = -x_1$; otherwise, $\lim_{z_{f\alpha} \to -\infty} \prod(x_1, y_1, w, m, x_2, y_2, z_{f\alpha}, z_{b\alpha}) = -\infty$. We partition $\Omega 2$ into the following two sets: $\Omega 2_A = \{(w, m): f_3(w, m)/f_2(w, m) \geq 1 > b_3(m)/b_2(m)\}$, $\Omega 2_B = \{(w, m): f_3(w, m)/f_2(w, m) > b_3(m)/b_2(m) \geq 1\}$. In $\Omega 2_A$, $y_2^* = 0$ due to $1 > b_3(m)/b_2(m)$. Thus,

$$\begin{aligned}\prod(x_1, y_1, w, m, x_2^*, y_2^*, z_{f\alpha}, z_{b\alpha} \mid (w, m) \in \Omega 2_A) &= [f_2(w, m) - 1]x_1 + [b_3(m) + z_{b\alpha} - 1]y_1 \\ &\geq [f_2(w_H, m_L) - 1]x_1 + [b_3(m_L) + z_{b\alpha} - 1]y_1 \\ &= \Pi(x_1, y_1, w_H, m_L, x_2^0, y_2^0, z_{f\alpha}, z_{b\alpha}) \geq -\beta \qquad (41)\end{aligned}$$

where the first inequality follows from the fact that $f_2(w_H, m_L)$ and $b_3(m_L)$ are the worst price realizations for $f_2(w, m)$ and $b_3(m)$, respectively, and the last inequality follows from Lemma A5.

We next show that $y_2^* = [B - x_1 - y_1 + f_2(w, m)x_1]/b_2(m)$ given that that $x_2^* = -x_1$ in $\Omega 2_B$: $\prod(x_1, y_1, w, m, x_2^*, y_2^*, z_{f\alpha}, z_{b\alpha} \mid (w, m) \in \Omega 2_B) = [f_2(w, m) - 1][1 + [b_3(m) + z_{b\alpha} - b_2(m)]/b_2(m)]x_1 + [b_3(m) + z_{b\alpha} - 1]y_1 + [[b_3(m) + z_{b\alpha} - b_2(m)]/b_2(m)][B - y_1]$ where

$[f_2(w, m) - 1][1 + [b_3(m) + z_{b\alpha} - b_2(m)]/b_2(m)]x_1 \geq 0$ (42)

following from $x_1 \geq 0$, $f_2(w, m) > 1$ (due to the definition of $\Omega 2_B$ and (5)), and $[b_3(m) + z_{b\alpha} - b_2(m)]/b_2(m) > -\beta/B > -1$ (due to the definition of $\Omega 2_B$, $\beta < B$, $z_{b\alpha} < 0$, (13));

$[b_3(m) + z_{b\alpha} - 1]y_1 > -\beta$ (43)

following from $y_1 \leq B$ and (13); and

$[[b_3(m) + z_{b\alpha} - b_2(m)]/b_2(m)][B - y_1] > -\beta$ (44)

following from $y_1 \leq B$ and $[b_3(m) + z_{b\alpha} - b_2(m)]/b_2(m) > -\beta/B > -1$ (due to the definition of $\Omega 2_B$, $\beta < B$, $z_{b\alpha} < 0$, (13)). Inequalities (42), (43) and (44) together imply that

$\prod(x_1, y_1, w, m, x_2^*, y_2^*, z_{f\alpha}, z_{b\alpha} \mid (w, m) \in \Omega 2_B) > -\beta$ (45)

where $x_2^* = -x_1$ and $y_2^* = [B - x_1 - y_1 + f_2(w, m)x_1]/b_2(m)$.

Following from (41), (45), and lemmas A6 and A8,

$$\left(x_2^*(x_1,y_1,w,m), y_2^*(x_1,y_1,w,m)\right) = \begin{cases} (-x_1, 0) & \text{if } (w,m) \in \Omega 1 \cup \Omega 2_A \\ \left(-x_1, \left(B - x_1 - y_1 + f_2(w,m)x_1\right)/b_2(m)\right) & \text{if } (w,m) \in \Omega 3 \cup \Omega 2_B \end{cases}.$$

Thus,

$$\partial E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^*, y_2^*, \tilde{z}_f, \tilde{z}_b)]/\partial y_1 = E[b_3(\tilde{m}) + \tilde{z}_b] - \iint_{\Omega 1 \cup \Omega 2_A} \phi_w(w)\phi_m(m)dwdm$$

$$- \iint_{\Omega 3 \cup \Omega 2_B} \left(b_3(m)/b_2(m)\right)\phi_w(w)\phi_m(m)dwdm$$

$$= E[b_3(\tilde{m}) + \tilde{z}_b] - 1 - V_c' \quad (46)$$

where $V_c' = \iint_{\Omega 3 \cup \Omega 2_B} \left(\frac{b_3(m)}{b_2(m)} - 1\right)\phi_w(w)\phi_m(m)dwdm$ which is nonnegative because the integrand is nonnegative by definitions of $\Omega 2_B$ and $\Omega 3$. Also, we have

$$\partial E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^*, y_2^*, \tilde{z}_f, \tilde{z}_b)]/\partial x_1 = E[f_3(\tilde{w}, \tilde{m}) + \tilde{z}_f] - \iint_{\Omega 1 \cup \Omega 2_A} \phi_w(w)\phi_m(m)dwdm$$

$$- \iint_{\Omega 3 \cup \Omega 2_B} \left(b_3(m)/b_2(m)\right)\phi_w(w)\phi_m(m)dwdm + \iint_{\Omega 1 \cup \Omega 2_A} \left(f_2(w,m) - f_3(w,m)\right)\phi_w(w)\phi_m(m)dwdm$$

$$+ \iint_{\Omega 3 \cup \Omega 2_B} \left(f_2(w,m)\frac{b_3(m)}{b_2(m)} - f_3(w,m)\right)\phi_w(w)\phi_m(m)dwdm$$

$$= E[f_3(\tilde{w}, \tilde{m}) + \tilde{z}_f] - 1 - V_c' + V_{l\cup s}' \quad (47)$$

where

$$V_{l\cup s}' = \iint_{\Omega 1 \cup \Omega 2_A} \left(f_2(w,m) - f_3(w,m)\right)\phi_w(w)\phi_m(m)dwdm + \iint_{\Omega 3 \cup \Omega 2_B} \left(f_2(w,m)\frac{b_3(m)}{b_2(m)} - f_3(w,m)\right)\phi_w(w)\phi_m(m)dwdm$$

Following from the definitions of $\Omega 2_B$ and $\Omega 3$, $w_\tau(m_\tau) = 0$ (see (18)), $E[\tilde{w}] = 0$, and the symmetry in $\phi_w(w)$, we have $\iint_{\Omega 3 \cup \Omega 2_B} \left(f_2(w,m)\frac{b_3(m)}{b_2(m)} - f_3(w,m)\right)\phi_w(w)\phi_m(m)dwdm = 0$. Following from the definitions of $\Omega 1$ and $\Omega 2_A$, $w_\tau(m) < 0$ for all $m < m_\tau$ (see (18)), $E[\tilde{w}] = 0$, and the symmetry in $\phi_w(w)$, we have $\iint_{\Omega 1 \cup \Omega 2_A} \left(f_2(w,m) - f_3(w,m)\right)\phi_w(w)\phi_m(m)dwdm \geq 0$. Thus, $V_{l\cup s}' \geq 0$. Following from (17),

$$\partial E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^*, y_2^*, \tilde{z}_f, \tilde{z}_b)]/\partial x_1 - \partial E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^*, y_2^*, \tilde{z}_f, \tilde{z}_b)]/\partial y_1 = V_{l\cup s}' \geq 0.$$

Moreover, following from the definitions of $\Omega 2$ and $\Omega 3$, $V_c' \leq V_c$ (see (32)). Recall that (21) implies (33) is positive (see the proof of Proposition 1). Thus, $V_c' \leq V_c$ implies that (46) is positive, i.e.,

$$\partial E[\Pi(x_1, y_1, \tilde{w}, \tilde{m}, x_2^*, y_2^*, \tilde{z}_f, \tilde{z}_b)]/\partial y_1 > 0. \ \square$$